Optimal Control of a Bilinear System with a Quadratic Cost Functional

1st Seda Goktepe Korpeoglu
Mathematical Engineering
Yildiz Technical University
Istanbul, Turkey
sgoktepe@yildiz.edu.tr

2nd Ismail Kucuk
Electrical and Electronics Engineering
Istanbul Sabahattin Zaim University
Istanbul, Turkey
ismail.kucuk@izu.edu.tr

***Abstract*—Various control systems in engineering problems are modelled with linear differential equations where a linear control is used. On the other hand, linear models are not capable of representing many systems where the control is applied in a multiplicative ways. These multiplicative controls yield bilinear systems (BLS). Products of state and control take part in BLS, which means that state and control are linear separately but not jointly. In this paper, optimal control of bilinear systems with a quadratic cost functional is studied. A distributed parameter system is considered and a bilinear control is applied to the system. The control problem is turned into a modal control problem by way of reduced order modelling. Performance index (cost functional) is defined as a measure of the dynamic response and a penalty term on control energy. Pontryagins maximum principle is used to obtain the optimal control function that leads to a nonlinear two-point boundary value problem. Optimal control and optimal trajectory of the system are determined by solving this two-point boundary value problem using steepest descent method. The programming is done in MATLAB platform. Numerical results are given in graphics for showing the effectiveness and applicability of the introduced bilinear control for a parabolic distributed parameter system.**

***Index Terms*—bilinear system, optimal control, parabolic equation, steepest descent method**

I. Introduction

Modelling of systems ranging from physical systems to social systems brings a nonlinearity to computations. Due to the fact that the linearization of nonlinear systems loses real features, developing new techniques and strategies are important to understand their natural properties and to improve their performance. Bilinear systems (BLS) are specific types of nonlinear systems: BLS include multiplication of state and control, that is, they are not linear together although linear on their own in state and control. In order to approximate and analyze complex nonlinear systems, BLS are used due to the simplicity of the system. Thus, modeling and control of nonlinear systems in a bilinear framework are essential problems in engineering [1]. Bilinear control processes are examined in three categories in the literature. Firstly, the concept of bilinear control that can be seen as a multiplicative control is applied to linear or nonlinear systems. Secondly, a bilinear control of distributed and lumped parameter systems with the concept of diffusivity-interior and boundary control is studied. For this type of control problem, a reduced order model is applied and the corresponding model is a bilinear system. Thirdly, an optimal control of bilinear systems with quadratic performance index is searched. Bilinear controls are used in many types of chemical and biological reactions as catalysts that can accelerate or decelerate the reaction [2]. Particle accelerators [3], nuclear power plants [4], plasma [5] and quantum devices [6] are some of the many research areas where control problems of bilinear systems are studied. The control of the rod equation and a schrödinger equation with bilinear control is simultaneously discussed by Kime [7]. Lin, Zhou and Gao [8] studied the bilinear control of linear parabolic system in the sense of exact controllability. Ouzahra, Tsouli and Boutoulout [9] examined the heat equation with bilinear control from the point of exact controllability and gave explicit control strategy with simulations. Kucuk, Sadek and Adali [10] studied active vibration control of plates integrated with distributed piezoelectric actuators by utilizing maximum principle. Sadek, Abualrub and El Nachar [11] considered optimal control of a bilinear parabolic equation. Wavelet based modal space expansion method is proposed to evaluate the optimal control and trajectory.

In this paper, we will focus on obtaining the optimal control and trajectories to a parabolic equation with bilinear control. The model problem studied is used to describe heat conduction process. The performance index to be minimized implies that the aim is to sustain the temperature close to its steady-state temperature value with a minimum expenditure of control effort. Optimal control laws are determined by applying Pontryagin's maximum principle. A nonlinear two-point boundary-value problem (TPBVP) with split boundary conditions results from this approach. The obtained TPBVP is solved by using steepest descent method via MATLAB. Numerical results are given to show the effectiveness and applicability of the introduced bilinear control for a parabolic distributed parameter system.

II. Construction of Optimal Control Problem

Consider a distributed parameter system describing a heat conduction process with a source parameter [12]

$$u_t = \Delta u + p(t)u + \phi(x,t), \tag{1}$$

978-1-5386-5257-2/18/$31.00 ©2018 IEEE

where $u = u(x,t) \in \Omega = \Omega_x \times \Omega_t$ is the displacement of the system at position x and time t. $p(t)$ is the control function to be determined optimally with external heat source $\phi(x,t)$. The set $\Omega_x = [0,\ell]$ is a subset of Euclidean space $\mathbb{R}^1$ and Ω_t denote a given time interval $(0, t_f)$ where t_f is a predetermined terminal time. (1) is subjected to the following initial and boundary conditions, respectively;

$$\begin{aligned} u(x,0) &= \varphi(x) & (2)\\ u(0,t) &= g_0(t), \quad u(\ell,t) = g_1(t) & (3) \end{aligned}$$

for given functions $g_0(t)$, $g_1(t)$ and $\varphi(x)$. Let the admissible control set be

$$U_{ad} = \{p(t) : p(t) \in L^2(\Omega_t)\}.$$

$H((0,\ell)) = L^2((0,\ell))$ is a Hilbert space such that

$$H((0,\ell)) = \{p(t) : (0,\ell) \to \mathbb{R}, \|p(t)\|^2 < \infty\}.$$

The performance index functional $\mathcal{J}[p(t)]$ is specified as a weighted quadratic functional of the dynamic response which is to be minimized at the terminal time t_f subject to (1),

$$\begin{aligned} \mathcal{J}[p(t)] &= \frac{1}{2}\int_{\Omega_x} [r_1 u^2(x,t_f) + r_2 u_t^2(x,t_f)]dx \\ &+ \frac{1}{2}\int_{\Omega} q u^2(x,t)dxdt + \int_{\Omega_t} \frac{1}{2} s p^2(t)dt, \quad (4) \end{aligned}$$

where r_1, r_2, q and s are weighting constants satisfying the condition $r_1, r_2, q \geq 0$ and $s > 0$. The last term of (4) is a penalty term which limits expending large amounts of control effort.

It is desired to find optimal control function $p^*(t) \in U_{ad}$ that minimizes the performance index

$$\mathcal{J}[p^*(t)] = \underset{p(t)\in U_{ad}}{min} \mathcal{J}[p(t)], \quad (5)$$

providing equations (1)-(3).

A. *Existence-Uniqueness of the solution and the control*

Energy methods that are used to research existence, uniqueness of a solution are important tools. The origin of these methods are the physical fact that the kinetic energy of a physical system expressed as a homogeneous equations of (1) decreases in time in the absence of external forcing [13]. The existence and uniqueness of the problem (1)-(3) are discussed in [14], [15], [16].

The uniqueness of the problem is proved in the following lemma by means of energy methods, because the uniqueness of solution of the system yields the uniqueness of control.

Lemma 1: The solution $u(x,t) \in L^2(\mathbb{R}^N)$ of (1) subject to (2)-(3) is unique.

Proof 1: Assume that there are two distinct smooth solutions u_1 and u_2 satisfying the (1). Then their difference, $\tilde{u} = u_1 - u_2$, satisfies the (1) with zero initial-boundary conditions, i.e,

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u} + p(t)\tilde{u}, & 0 \leq x \leq \ell, \quad 0 < t < t_f \\ \tilde{u}(x,0) = 0, \\ \tilde{u}(0,t) = 0, \quad \tilde{u}(\ell,t) = 0. \end{cases} \quad (6)$$

Now define the "energy" integral as follows

$$E[\tilde{u}] = \frac{1}{2}\int_0^\ell |\tilde{u}(x,t)|^2 dx \quad (7)$$

which is always positive and decreasing, if $\tilde{u}$ solves (1). The energy arises if one takes inner product $(.,.)$ of $\tilde{u}_t$ with $\tilde{u}$ and integrates with respect to x over the spatial interval $[0,\ell]$.
Take the L^2 inner product $(.,.)$ of this equation with $\tilde{u}$ to get

$$(\tilde{u}_t, \tilde{u}) = (\Delta\tilde{u}, \tilde{u}) + (p(t)\tilde{u}, \tilde{u}), \quad (8)$$

Let $||.||_0$ denote the L^2 norm on $\mathbb{R}^N$ given by

$$||f||_0 = \left(\int_{\mathbb{R}^N} |f|^2 dx\right)^{1/2}.$$

Differentiating with respect to t (7),

$$\begin{aligned} \frac{d}{dt}E &= \int_0^\ell \tilde{u}\tilde{u}_t dx = \int_0^\ell \tilde{u}(\tilde{u}_{xx} + p(t)\tilde{u})dx \\ &= \int_0^\ell \tilde{u}\tilde{u}_{xx}dx + \int_0^\ell p(t)\tilde{u}^2 dx. \quad (9) \end{aligned}$$

Using integrating by parts in the last integral leads to

$$\frac{d}{dt}E = \tilde{u}\tilde{u}_x|_0^\ell - \int_0^\ell \tilde{u}_x^2 dx + \int_0^\ell p(t)\tilde{u}^2 dx \quad (10)$$

we get

$$\begin{aligned} \frac{d}{dt}E &= -\int_0^\ell \tilde{u}_x^2 dx + \int_0^\ell p(t)\tilde{u}^2 dx \\ &\leq p(t)\|\tilde{u}\|_0^2. \end{aligned}$$

Then, we get

$$\frac{d}{dt}\|\tilde{u}\|_0 \leq p(t)\|\tilde{u}\|_0.$$

We now apply the Gronwall's lemma to obtain

$$\|u_1 - u_2\|_0 \leq \|u_1 - u_2|_{t=0}\|_0 exp\left(\int_0^{t_f} p(r)dr\right).$$

As an immediate consequence we obtain the uniqueness of smooth solutions.
The control function $p(t)$ is unique to preserve the uniqueness of the solution provided by **Lemma 1**. The system (1)-(3) is observable because the system has a unique solution and control function. By taking Hilbert Uniqueness into account, the observability is equivalent to the controllability [17], [18]. Briefly, the system is controllable.

III. Modal Control Space Problem

In the present section, using reduced order modeling, modal space expansion, the distributed parameter system (1)-(5) is transformed into lumped parameter system. New system gives rise to a bilinear system. In order to achieve the transformation, first a new parameter $w(x,t)$ is introduced to get homogeneous boundary conditions.
By letting

$$w(x,t) = u(x,t) - \frac{(\ell - x)}{\ell}g_0(t) - \frac{x}{\ell}g_1(t), \quad (11)$$

in (1), the following new distributed parameter system is obtained

$$\begin{aligned} w_t - w_{xx} &= \frac{(x-\ell)}{\ell} g_{0t} - \frac{x}{\ell} g_{1t} + p(t)\big(w + \frac{(\ell - x)}{\ell} g_0 \\ &+ \frac{x}{\ell} g_1\big) + \phi(x,t), \end{aligned} \tag{12}$$

subject to

$$w(x,0) = \varphi(x) - \frac{(\ell-x)}{\ell} g_0(0) - \frac{x}{\ell} g_1(0), \tag{13}$$

$$w(0,t) = 0, \quad w(\ell,t) = 0. \tag{14}$$

Theorem 1: Any $w(x,t) \in H(0,\ell)$ has a unique representation [19]

$$w(x,t) = \sum_{n=1}^{\infty} \psi_n(x) y_n(t) \tag{15}$$

where $\{\psi_n(x)\}_{n=1}^{\infty}$ is a complete orthonormal basis in $H(0,\ell)$ and $y_n(t)$ is the temporal term.

Having a modal space expansion gives rise to an infinite-dimensional system theoretically which makes the problem physically insurmountable since there will be a large number of modes to control. Hence, a truncated Fourier series expansion of (15) is taken in the computations hereafter

$$w(x,t) \approx \sum_{n=1}^{N} \psi_n(x) y_n(t). \tag{16}$$

Denoting a complete orthonormal basis as

$$V = \big\{ v | v, \frac{\partial v}{\partial x} \in H((0,\ell)) \quad \text{and} \quad v|_{\partial(0,\ell)} = 0 \big\} \tag{17}$$

and by multiplying both sides of (12) by a basis function v, the solution $w(x,t)$ of the system satisfies

$$\begin{aligned} \int_0^\ell \frac{\partial w}{\partial t} v dx - \int_0^\ell \frac{\partial^2 w}{\partial x^2} v dx &= \int_0^\ell (\frac{x-\ell}{\ell} g_{0t} - \frac{x}{\ell} g_{1t}) v dx \\ &+ \int_0^\ell p(w + \frac{\ell - x}{\ell} g_0 + \frac{x}{\ell} g_1) v dx \\ &+ \int_0^\ell \phi(x,t) v dx \end{aligned}$$

where $w, v \in \mathcal{V}$. If we use expression (16) for $w(x,t)$ and $v = \psi_m$, $1 \le m \le \infty$ and using the following notations,we get the finite dimensional system ($D \triangleq \frac{\partial}{\partial x}$):

$$M_{mn} = (\psi_m, \psi_n) = \int_0^\ell \psi_m(x)\psi_n(x) dx = \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

$$K_{mn} = (\psi_m, D^2\psi_n) = \int_0^\ell \psi_m(x) \frac{\partial^2(\psi_n(x))}{\partial x^2} dx \tag{18}$$

$$F_m = \int_0^\ell (\frac{(x-\ell)}{\ell} g_{0t} - \frac{x}{\ell} g_{1t}) \psi_m(x) dx$$

$$G_m = \int_0^\ell (\frac{(\ell-x)}{\ell} g_0 - \frac{x}{\ell} g_1) \psi_m(x) dx \tag{19}$$

$$H_m = \int_0^\ell \phi(x,t)\psi_m(x) dx$$

$$L_m = F_m + H_m, \tag{20}$$

$$\frac{dz}{dt} = Kz + pz + Gp + L \tag{21}$$

where $z(t) = (y_1(t), y_2(t), ..., y_n(t))^T \in \mathbb{R}^N$, $M, K \in \mathbb{R}^{N\times N}$, $G, L \in \mathbb{R}^N$ and $p = p(t)$ is control vector. The vector $z(t)$ is the finite dimensional approximation. The initial values are determined by

$$y_n(0) = (w(x,0), \psi_m), \tag{22}$$

$m, n = 1, 2, ..., N$.

IV. Derivation of the Pontryagin's Maximum Principle for the Bilinear System

Theorem 2: If we apply the Pontryagin's Maximum Principle to bilinear system in (21), a canonical optimality condition is obtained,

$$\begin{cases} \dot{z}(t) = Kz(t) - s^{-1}\Lambda^T(t)(z(t)+G)^2 + L \\ \dot{\Lambda}(t) = -Qz(t) - K^T\Lambda(t) - s^{-1}(z(t)+G)^T\Lambda^2(t) \\ \Lambda(t_f) = R_1 z(t_f) + R_2 \dot{z}(t_f) \\ z(t_0) = z_0 \end{cases} \tag{23}$$

which is a nonlinear two-point boundary value problem (TPBVP). In (23), Q, R_1 and R_2 are positive semidefinite symmetric $n \times n$ matrices, K, G and L are defined in (18), (19) and (20), respectively.

Proof 2: Consider the optimal control problem of the bilinear system (21)-(22)

$$\frac{dz}{dt} = Kz + pz + Gp + L,$$

where $z(t)$ is the finite dimensional approximation of $w(x,t)$ and $p(t)$ is the control input. K, G and L are defined in (18), (19) and (20), respectively. The quadratic cost functional associated with (21) is given by

$$\begin{aligned} \min_p \mathcal{J} &= \frac{1}{2}[z(t_f)]^T R_1 [z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2 [\dot{z}(t_f)] \\ &+ \frac{1}{2}\int_{t_0}^{t_f} z(t)^T Q z(t) dt + \frac{1}{2}\int_{t_0}^{t_f} s p^2(t) dt. \end{aligned}$$

where $R_{1mn} = \int_0^\ell r_1 \psi_m(x)\psi_n(x) dx$, $R_{2mn} = \int_0^\ell r_2 \psi_m(x)\psi_n(x) dx$ and $Q_{mn} = \int_0^\ell q \psi_m(x)\psi_m(x) dx$ respectively, for $m, n = 1, ..., N$.

Using the augmented cost functional, we minimize the cost functional

$$\begin{aligned}\mathcal{J}^*[z,p,\Lambda] &= \int_{t_0}^{t_f}\{\frac{1}{2}(z^TQz+sp^2) \quad (24)\\ &- \Lambda^T(t)(\dot{z}(t)-Kz(t)-p(t)z(t)-Gp-L)\}dt\\ &+ \frac{1}{2}[z(t_f)]^TR_1[z(t_f)]+\frac{1}{2}[\dot{z}(t_f)]^TR_2[\dot{z}(t_f)]\end{aligned}$$

Introducing the so-called Hamiltonian, the augmented functional becomes

$$\begin{aligned}\mathcal{J}^*[z,p,\Lambda] &= \int_{t_0}^{t_f}[\mathcal{H}(t,z,p,\Lambda)-\Lambda^T\dot{z}(t)]dt \quad (25)\\ &+ \frac{1}{2}[z(t_f)]^TR_1[z(t_f)]+\frac{1}{2}[\dot{z}(t_f)]^TR_2[\dot{z}(t_f)].\end{aligned}$$

If (z,p,Λ) is a minimizer of $\tilde{\mathcal{J}}$

$$\begin{aligned}\delta\mathcal{J}^* &= \int_{t_0}^{t_f}[\frac{\partial\mathcal{H}}{\partial z}\delta z+\frac{\partial\mathcal{H}}{\partial p}\delta p+\frac{\partial\mathcal{H}}{\partial\Lambda}\delta\Lambda-\delta(\Lambda^T(t)\dot{z}(t))]dt\\ &+ \delta(\frac{1}{2}[z(t_f)]^TR_1[z(t_f)]+\frac{1}{2}[\dot{z}(t_f)]^TR_2[\dot{z}(t_f)])=0,\end{aligned}$$

After processing of variation operations and using integrating by parts we obtain firstly,

$$\begin{aligned}\mathcal{H}_p &= 0\\ p(t) &= -s^{-1}\Lambda^T(t)(z+G)\end{aligned}$$

Secondly,

$$\begin{aligned}\mathcal{H}_z+\dot{\Lambda}^T &= 0\\ \dot{\Lambda}(t) &= -Qz(t)-K^T\Lambda(t)-p^T\Lambda(t)\end{aligned}$$

Thirdly,

$$\begin{aligned}\mathcal{H}_\Lambda-\dot{z}(t) &= 0\\ \dot{z}(t) &= Kz(t)-s^{-1}\Lambda^T(t)(z(t)+G)^2+L\end{aligned}$$

Lastly,

$$\begin{aligned}[z(t_f)]^TR_1+[\dot{z}(t_f)]^TR_2-\Lambda^T(t_f) &= 0\\ R_1z(t_f)+R_2\dot{z}(t_f) &= \Lambda(t_f)\end{aligned}$$

In this proof, Pontryagin's maximum principle leads to a nonlinear TPBVP that cannot be solved analytically to obtain the control law. The difficulty of solving this optimal control problem is caused by the combination of split boundary values and nonlinear differential equations.

V. Steepest Descent Method for Determining Optimal Controls and Trajectories

In this part, we will discuss one of iterative numerical techniques for determining optimal controls and trajectories. Steepest descent method is a procedure for solving nonlinear two-point boundary-value problems [20].

Suppose that a nominal control $p^{(i)}(t)$, $t\in[t_0,t_f]$, is known and used to solve the differential equations

$$\dot{z}^{(i)}(t) = a(z^{(i)}(t),p^{(i)}(t),t) \quad (26)$$

$$\dot{\Lambda}^{(i)}(t) = -\frac{\partial\mathcal{H}}{\partial z}(z^{(i)}(t),p^{(i)}(t),\Lambda^{(i)}(t),t) \quad (27)$$

so that the nominal state-costate trajectory $z^{(i)}$, $\Lambda^{(i)}$ satisfies the boundary conditions

$$z^{(i)}(t_0) = z_0 \quad (28)$$

$$\Lambda^{(i)}(t_f) = \frac{\partial h}{\partial z}(z^{(i)}(t_f)). \quad (29)$$

If this nominal control also satisfies

$$\frac{\partial\mathcal{H}}{\partial p}(z^{(i)}(t),p^{(i)}(t),\Lambda^{(i)}(t),t)=0, \quad (30)$$

then $p^{(i)}(t)$, $z^{(i)}(t)$, and $\Lambda^{(i)}(t)$ are extremal. Suppose that (30) is not satisfied, the variation of the augmented functional $\mathcal{J}$ on the nominal state-costate-control history is

$$\begin{aligned}\delta\mathcal{J} &= [\frac{\partial h}{\partial z}(z^{(i)}(t_f))-\Lambda^{(i)}(t_f)]^T\delta z(t_f)\\ &+ \int_{t_0}^{t_f}\{[\dot{\Lambda}^{(i)}(t)+\frac{\partial\mathcal{H}}{\partial z}(z^{(i)}(t),p^{(i)}(t),\Lambda^{(i)}(t),t)]^T\delta z(t)\\ &+ [\frac{\partial\mathcal{H}}{\partial p}(z^{(i)}(t),p^{(i)}(t),\Lambda^{(i)}(t),t)]^T\delta p(t)\\ &+ [a(z^{(i)}(t),p^{(i)}(t),t)-\dot{z}^{(i)}(t)]^T\delta\Lambda(t)\}dt\end{aligned}$$

where $\delta z(t)\triangleq z^{(i+1)}(t)-z^{(i)}(t)$, $\delta p(t)\triangleq p^{(i+1)}(t)-p^{(i)}(t)$ and $\delta\Lambda(t)\triangleq\Lambda^{(i+1)}(t)-\Lambda^{(i)}(t)$.

If (26) through (29) are satisfied, then

$$\delta\mathcal{J}=\int_{t_0}^{t_f}[\frac{\partial\mathcal{H}}{\partial p}(z^{(i)}(t),p^{(i)}(t),\Lambda^{(i)}(t),t)]^T\delta p(t)dt.$$

Since $\delta\mathcal{J}$ is the linear part of the increment $\Delta\mathcal{J}\triangleq\mathcal{J}(p^{(i+1)})-\mathcal{J}(p^{(i)})$ and the norm of δp, $\|p^{(i+1)}-p^{(i)}\|$, is small, the sign of $\Delta\mathcal{J}$ will be determined by the sign of $\delta\mathcal{J}$. Since our goal is to minimize $\mathcal{J}$, we wish to make $\Delta\mathcal{J}$ negative. If we select the change in p as

$$\delta p(t)=p^{(i+1)}(t)-p^{(i)}(t)=-\tau\frac{\partial\mathcal{H}^{(i)}}{\partial p}(t),$$

with $\tau>0$, then

$$\delta\mathcal{J}=-\tau\int_{t_0}^{t_f}[\frac{\partial\mathcal{H}^{(i)}}{\partial p}(t)]^T[\frac{\partial\mathcal{H}^{(i)}}{\partial p}(t)]dt\leq 0,$$

because the integrand is nonnegative for all $t\in[t_0,t_f]$. The inequality holds if and only if

$$\frac{\partial\mathcal{H}^{(i)}}{\partial p}(t)=0,$$

for all $t\in[t_0,t_f]$.

A. Simulation Study

Let consider the following controlled model

$$u_t=u_{xx}+p(t)u+e^t\sin(\pi x)$$

subject to

$$\begin{aligned}u(x,0) &= \sin(\pi x)\\ u(0,t) &= 0,\quad u(1,t)=0.\end{aligned}$$

The solution of the system composed by the nonlinear TPBVP and proposed bilinear control is performed by using a finite dimensional approximation $z(t)$ of $w(x,t)$ based on modal space expansion technique. The state equation can be expressed as a series of orthonormal basis functions $w(x,t) \approx \sum_{n=1}^{N} \psi_n(x) y_n(t)$ where $\psi_n(x) = \sqrt{2}\sin(n\pi x)$, which satisfy the homogeneous boundary conditions. We obtain a finite dimensional system

$$\dot{z}(t) = Kz(t) + p(t)z(t) + Gp(t) + L$$

with initial condition z(0)=0. The performance index to be minimized is

$$\begin{aligned} \mathcal{J} &= \frac{1}{2}[z(t_f)]^T R_1 [z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2 [\dot{z}(t_f)] \\ &+ \frac{1}{2}\int_{t_0}^{t_f} z^T Q z dt + \frac{1}{2}\int_{t_0}^{t_f} sp^2(t) dt, \end{aligned}$$

where the elements of R_1, R_2 and Q are weighting factors that defined by $R_{1mn} = \int_0^1 r_1 \sin(m\pi x)\sin(n\pi x)dx$, $R_{2mn} = \int_0^1 r_2 \sin(m\pi x)\sin(n\pi x)dx$ and $Q_{mn} = \int_0^1 q \sin(m\pi x)\sin(n\pi x)dx$ respectively, for $m, n = 1, ..., N$. We simulate the system (1)-(3) over $t_0 = 0 \le t \le t_f = 1$ with following parameters: $\phi(x,t) = e^t \sin(\pi x)$, $n = m = 1$, $\ell = 1$, $\psi_n(x) = \sqrt{2}\sin(n\pi x)$, $\psi_m(x) = \sqrt{2}\sin(m\pi x)$.
We select s arbitrarily as 0.1, $r_1 = 0.1$, $r_2 = 0.1$ and $q = 1$. The costate equation is determined from the Hamiltonian,

$$\begin{aligned} \mathcal{H}(t, z, p, \Lambda) &= \frac{1}{2}(z(t)^T Q z(t) + sp^2(t)) \\ &- \Lambda^T(t)(-Kz(t) - p(t)z(t) - Gp(t) - L) \end{aligned}$$

as

$$\dot{\Lambda}^T = -\frac{\partial \mathcal{H}}{\partial z} = -Qz(t) - K^T\Lambda(t) - s^{-1}(z(t) + G)^T \Lambda^2(t)$$

The algebraic relation that must be satisfied is

$$\frac{\partial \mathcal{H}}{\partial P} = sp(t) + \Lambda^T(t)[z(t) + G] = 0.$$

The boundary condition at $t = t_f = 1$ is $\Lambda(1) = R_1 z(1) + R_2 \dot{z}(1)$.

MATLAB was used for the solution of the system. Numerical solution was carried out using a fourth-order Runge-Kutta method and an integration interval of 0.01 unit. The norm used was

$$\|\frac{\partial \mathcal{H}}{\partial p}\|^2 = \int_0^1 [\frac{\partial \mathcal{H}}{\partial p}(t)]^2 dt,$$

and the iterative procedure was stopped when $\|\frac{\partial \mathcal{H}}{\partial p}\| \le 10^{-3}$. Simulation of the uncontrolled PDE system (1) is shown in Fig. 1. With $p^0(t) = 1.0, \quad t \in [0,1]$ and an initial $\tau = 0.4$, the value of the performance index as a function of iteration numbers is plotted in Fig. 2. After 27 iterations, the performance index reduces significantly; however, the rest 53 iterations yield very slight improvement. This type of progress is typical of the steepest descent method. Spatial temperature profile of the controlled/uncontrolled of the bilinear system at the terminal time $t_f = 1$ is shown in Fig. 3. The temperature profiles in time for fixed value of $x_0 = 0.5$ for controlled/uncontrolled of the bilinear system is shown in Fig. 4. The simulation of the controlled PDE system is plotted in Fig. 5.

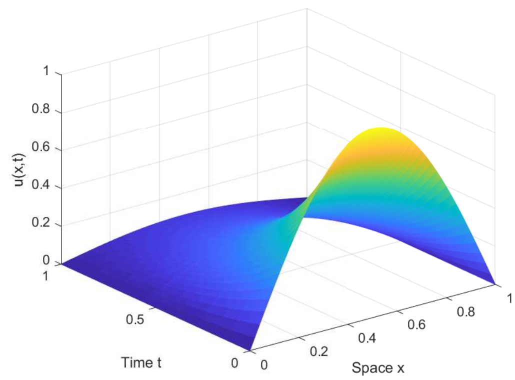

Fig. 1. Uncontrolled dynamics of system (1)-(3).

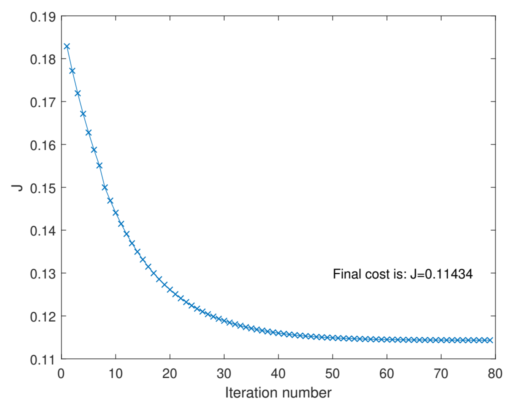

Fig. 2. Performance index reduction

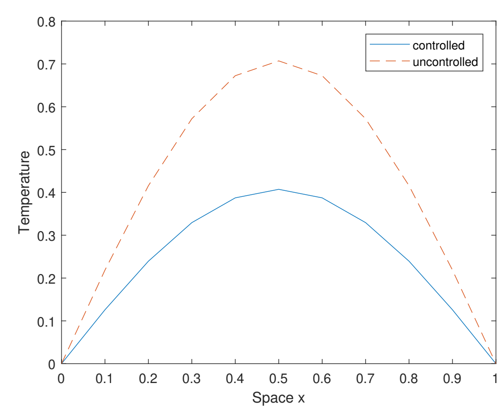

Fig. 3. Comparison of the spatial temperature profile at $t_f = 1$

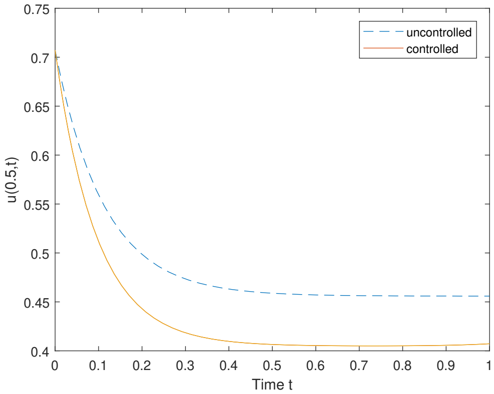

Fig. 4. Comparison of the temperature profiles in time at $x_0 = 0.5$

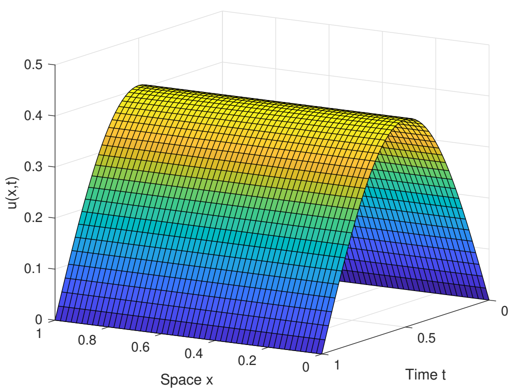

Fig. 5. Controlled PDE system simulation at $x_0 = 0.5$

VI. Conclusion

In this study, we discuss an optimal control problem of a parabolic PDE system with bilinear control by means of Pontryagin's Maximum Principle. Wellposedness and contollability of the solution to the system is considered. The optimal control problem is reduced to solve a nonlinear two-point boundary value problem by using maximum principle. The difficulty of solving this reduced differential equations involving state and costate equations is caused by combination of nonlinear equations and split boundary values. Steepest descent method is choosen as a solution technique for TPBVP. The programming is done in `MATLAB` platform. Numerical results are given in graphics for showing the effectiveness and applicability of the introduced bilinear control for a parabolic distributed parameter system.

References

[1] B. S. Kim and M. T. Lim, "Robust H∞ control method for bilinear systems," *Int. J. Control Autom. Syst.*, vol. 1, no. 2, pp. 171–177, 2003.

[2] A. Y. Khapalov, "Controllability of the semilinear parabolic equation governed by a multiplicative control in the reaction term: A qualitative approach," *SIAM J. Control Optim.*, vol. 41, no. 6, pp. 1886–1900, 2003.

[3] H. Modares, A. Alfi, and M.-B. N. Sistani, "Parameter estimation of bilinear systems based on an adaptive particle swarm optimization," *Eng. Appl. Artif. Intell.*, vol. 23, no. 7, pp. 1105 – 1111, 2010.

[4] M. Ekman, "Modeling and control of bilinear systems: Application to the activated sludge process," Ph.D. dissertation, Uppsala University, Sweden, 2005.

[5] Y. Ou, "Optimal control of a class of nonlinear parabolic pde systems arising in fusion plasma current profile dynamics," Ph.D. dissertation, Lehigh University, USA, 2010.

[6] A. G. Butkovskiy and Y. I. Samoilenko, *Control of Quantum-Mechanical Processes and Systems Mathematics and Its Applications*. Mathematics and its Applications Soviet Series, Kluwer academic publishers Group, Dordrecht, 1990, vol. 56.

[7] K. A. Kime, "Simultaneous control of a rod equation and a simple schrödinger equation," *Syst. Control Lett.*, vol. 24, no. 4, pp. 301–306, 1995.

[8] P. Lin, Z. Zhou, and H. Gao, "Exact controllability of the parabolic system with bilinear control," *Appl. Math. Lett.*, vol. 19, no. 6, pp. 568–575, 2006.

[9] M. Ouzahra, A. Tsouli, and A. Boutoulout, "Exact controllability of the heat equation with bilinear control," *Math. Method Appl. Sci.*, vol. 38, no. 18, pp. 5074–5084, 2015.

[10] I. Sadek, I. Kucuk, and S. Adali, "Active open-loop control of plates with multiple piezoelectric patches via the maximum principle," *Mech. Adv. Mater. Struc.*, vol. 21, no. 9, pp. 772–779, 2014.

[11] T. Abualrub, I. Sadek, and F. E. Nachar, "Wavelet based approximations in the optimal control of parabolic problems," *J. Control Theory Appl.*, vol. 11, no. 1, pp. 103–107, 2013.

[12] M. Tatari and M. Dehghan, "Identifying a control function in parabolic partial differential equations from overspecified boundary data," *Comput. Math. Appl.*, vol. 53, no. 12, pp. 1933–1942, 2007.

[13] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*. Cambridge University Press, 2002, vol. 27.

[14] J. R. Cannon and Y. Lin, "Determination of a parameter p(t) in some quasi-linear parabolic differential equations," *Inverse Probl.*, vol. 4, no. 1, p. 35, 1988.

[15] K. Deckert and C. Maple, "Solution for diffusion with integral type boundary conditions," in *The Proceedings of the Iowa Academy of Science*, vol. 70, 1963, pp. 354–361.

[16] A. I. Prilepko and V. Solovev, "Solvability of the inverse boundary-value problem of finding a coefficient of a lower-order derivative in a parabolic equation," *J. Differential Equations*, vol. 23, no. 1, pp. 101–107, 1987.

[17] H. F. Guliyev and K. S. Jabbarova, "The exact conrollability problem for the second order linear hyperbolic equation," *J. Difference Equ. Appl.*, no. 3, pp. 10–19, 2010.

[18] M. Pedersen, *Functional Analysis in Applied Mathematics and Engineering*. CRC press, 1999, vol. 31.

[19] F. Riesz, "Sur les systèmes orthogonaux de fonctions," *CR Acad. Sci. Paris*, vol. 144, pp. 615–619, 1907.

[20] D. E. Kirk, *Optimal Control Theory: An Introduction*. Courier Corporation, 2012.