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δ-primary subhypermodules on Krasner hyperrings

Elif Kaya, Melis Bolat, Bayram Ali Ersoy, Serkan Onar, Kostaq Hila*, and Bijan Davvaz

Abstract. In this paper, we study commutative Krasner hyperrings with nonzero identity and nonzero unital hypermodules. We introduce a new concept, the δ-primary subhypermodule on Krasner hyperrings. Some characterizations and properties for δ-primary subhypermodules using the expansion function δ are provided. The images and inverse images of δ-primary subhypermodules under homomorphism are investigated. Finally, some characterizations for multiplication hypermodules with some special conditions are provided.

1 Introduction

The hyperstructure theory represents a generalization of algebraic structures. In algebra, the operation ensures that the element of one set corresponds to the value of an element of the other set. In hyperstructures,

* Corresponding author
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the hyperoperation assigns more than one element to a member of another set. The class of structures consisting of a non-empty set and at least one hyperoperation is called algebraic hyperstructures. Hyperalgebra is also a kind of hyperstructure, such as hypergroups, hyperrings, and hypermodules. In other words, hypergroup, hyperring and hypermodule structures are the generalized forms of groups, rings, and modules, respectively, through hyperoperation. Hyperstructures, concretely hypergroups, were introduced by Marty in 1934 [17]. Mittas pioneered the theory of canonical hypergroups in [19]. Krasner initiated hyperrings and hyperfields in 1983 [16]. Krasner's students, Mittas and Stratigopoulos, pioneered hypermodules, and Massouros also worked on free and cyclic hypermodules [18], [19], [20], [22]. Also, Ameri and Norouzi, Corsini, Dasgupta, Davvaz, Omidi, and Leoreanu Fotea studied hyperrings, in more detail in [3], [6], [7], [9], [8]. Overall, the study of hyperstructures has attracted significant attention from mathematicians worldwide, leading to a rich body of research in various areas of hyperstructure theory. The applications of hyperstructures extend beyond mathematics into fields such as computer science, physics, and cryptography.

Hyperstructure theory is actively used in computer sciences, mathematics, geometry, logic theory, and other domains inside and outside of mathematics. The brief history of hyperstructures, their usage patterns, the connections between each other, and their semantic infrastructure are examined in Golzio's article [12]. The article delves into the evolution of hyperstructure theory and its impact on different disciplines, shedding light on the interdisciplinary nature of this mathematical concept. Golzio's comprehensive analysis provides valuable insights into the practical implications and potential future developments of hyperstructures in various fields.

Prime and primary ideals have an essential place and a key role in algebra. For this reason, many researchers have worked in this field. Looking at the respective studies, Dongsheng defined δ-primary ideals, which unify prime and primary ideals, in 2001 [24]. δ-primary ideals are also extensions of prime ideals. The concept of δ-primary ideals provides a more comprehensive understanding of the relationships between prime and primary ideals in algebra. By unifying these two types of ideals, Dongsheng's work has opened up new avenues for research and exploration in the field. Ozel Ay, Yesilot, and Abumghaiseeb, Ersoy applied δ-primary ideals to hyperstructures in [2] and [21], respectively. In 2018, Yesilot et al. studied δ-primary modules [23]. Yetkin [5] focused on 2-absorbing δ-semiprimary ideals in commutative rings, a concept that combines the properties of 2-absorbing ideals and δ-semiprimary ideals to study the structure of rings in a more nuanced way. Ersoy et al. [11] defined the theory of ϕ-δ-primary submodules in module theory, which builds upon the concept of ϕ-primary submodules to provide a deeper understanding of module structures. Dawwas [1] introduced the concept of graded δ-primary structures in algebraic structures, which extends the idea of graded primary structures to incorporate the notion of δ-primary structures, offering a more comprehensive framework for studying graded objects. Badavi [4] studied the notion of weakly 2-absorbing δ-primary ideals in commutative rings, which further refines the concept of 2-absorbing primary ideals by incorporating the δ-primary property. Abdelhaq et al. in [10] investigated 1-absorbing δ-primary ideals in commutative rings, building upon the work of Badawi to explore a more specific case of δ-primary ideals. Hamoda [13] introduced the concept of weakly (m, n)-closed δ-primary ideals and further explored their properties and relationships with other types of ideals in commutative rings. Jabera [14] gave the notion of ϕ-$S-1$-absorbing δ-primary ideals and further explores their properties and relationships with other types of ideals in commutative rings. Then, he suggested [15] the idea of ϕ-$S-1$-absorbing δ-primary superideals over commutative super-rings in order to generalize the concept to a broader algebraic structure. This extension opens up new avenues for research in super-ring theory and expands the study of primary ideals to a wider class of rings. This research provides a deeper understanding of the behavior of these ideals and their significance in ring theory. These studies have expanded the understanding of prime and primary ideals in algebra, providing new insights into their applications. The concept of δ-primary ideals continues to be a topic of interest for researchers exploring the connections between algebraic structures and hyperstructures.

In this paper, we investigate a new concept, "δ-primary subhypermodules on Krasner hyperrings". Throughout the study, we focus on commutative Krasner hyperrings with nonzero units and nonzero unital hypermodules. $(\Re, \oplus, \circ)$ will be a commutative Krasner hyperring with a nonzero identity. We denote the set of all hyperideals of $\Re$ by $L(\Re)$ and the set of all proper hyperideals of $\Re$ by $L^*(\Re)$. Firstly, we define δ-primary subhypermodule on Krasner hyperrings, and we provide some characterizations for δ-primary subhypermodules. Then we investigate whether the union of the collection of δ-primary subhypermodules preserves the algebraic structure. Besides, we examine the images and inverse images of δ-primary subhypermodules under homomorphism. Finally, we provide some characterizations for multiplication hypermodules with some special conditions.

2 Preliminaries

Marty [17] defined hyperstructures, hypergroupoid, hypergroup, subhypergroup and commutative hypergroup in 1934. Then Mittas [19] innovated canonical hypergroups and Krasner [16] established Krasner hyperrings and hyperfields. In this part, we will give some basic definitions and theorems related to hyperstructures and their properties, building upon the foundational work of Marty, Mittas, and Krasner. Understanding these concepts is crucial for further exploration of the applications and implications of hyperstructures in various mathematical contexts.

Definition 2.1. [17] Let $\Re$ be a non-empty set, $P^*(\Re)$ represents the family of non-empty subsets of $\Re$ and $\circ : \Re \times \Re \longrightarrow P^*(\Re)$ is a hyperoperation. Let $(\Re, \circ)$ be a hypergroupoid.

i) $\forall a, b, c \in \Re$, if $a \circ (b \circ c) = (a \circ b) \circ c$, which means, $\bigcup\limits_{u \in a \circ b} u \circ c = \bigcup\limits_{v \in b \circ c} a \circ v$ then $\Re$ is called a semihypergroup.

ii) $\forall a \in \Re$, if there exists $e \in \Re$ such that $a \in (e \circ a) \cap (a \circ e)$ in another phrase $\{a\} \subseteq (e \circ a) \cap (a \circ e)$, then e is called identity element.

Definition 2.2. Let $(\Re, \circ)$ be a semihypergroup. $\forall a \in \Re$, if $a \circ \Re = \Re \circ a = \Re$, then $(\Re, \circ)$ is called a hypergroup.

Definition 2.3. Let $(\Re, \circ)$ be a hypergroup and $K \neq \emptyset$ be a subset of $\Re$. If $a \circ K = K \circ a = K$, $\forall a \in K$, then $(K, \circ)$ is called a subhypergroup of $(\Re, \circ)$.

Definition 2.4. Let $(\Re, \circ)$ be a hypergroup. If $a \circ b = b \circ a$, $\forall a, b \in \Re$, then $(\Re, \circ)$ is a commutative hypergroup.

Definition 2.5. [19] Let $\Re \neq \emptyset$. $(\Re, \oplus)$ is called a canonical hypergroup ($\oplus$ is a hyperoperation) if the following axioms are satisfied:

i) $a \oplus b \subseteq \Re$, for $a, b \in \Re$,

ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, for $a, b, c \in \Re$,
iii) $a \oplus b = b \oplus a$, for $a, b \in \Re$,
iv) $\exists 0 \in \Re$ such that $a \oplus 0 = \{a\}$, for any $a \in \Re$,
v) $\exists! a' \in \Re$, such that $0 \in a \oplus a'$, for any $a \in \Re$,
vi) $c \in a \oplus b$ implies that $b \in (-a) \oplus c$ and $a \in c \oplus (-b)$, that means $(\Re, \oplus)$ is reversible.

In this paper, we denote $c \oplus (-b)$ with $c \ominus b$.

Definition 2.6. [16] $(\Re, \oplus, \circ)$ is called a Krasner hyperring if the following statements hold:
i) $(\Re, \oplus)$ is a canonical hypergroup;
ii) $(\Re, \circ)$ is a semigroup having 0 as $a \circ 0 = 0 \circ a = 0$, for all $a \in \Re$;
iii) $(b \oplus c) \circ a = (b \circ a) \oplus (c \circ a)$ and $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$, for all $a, b, c \in \Re$.

Definition 2.7. [18] Let $(\Re, \oplus, \circ)$ be a hyperring with unit element 1. $(M, +)$ is a commutative hypergroup with the map $\cdot : \Re \times M \to P^*(M)$ defined by $(a, m) \to a \cdot m \in M$, " $+$ " is a hyperoperation and " $\cdot$ " is an external hyperoperation. Then M is called an $\Re$-hypermodule if the following statements hold, for all $a, b \in \Re$ and $m, n \in M$:
i) $(a \oplus b) \cdot m = a \cdot m + b \cdot m$,
ii) $a \cdot (m + n) = a \cdot m + a \cdot n$,
iii) $(a \circ b) \cdot m = a \cdot (b \cdot m)$,
iv) $a \cdot 0_M = 0_M$,
v) $1 \cdot m = m$.

Definition 2.8. [22] Let M be an $\Re$-hypermodule. Then M is said to be a multiplication $\Re$-hypermodule if for all subhypermodule N of M, there exists a hyperideal I of $\Re$ such that $N = I \circ M$.

Definition 2.9. [21, 24] δ-primary hyperideal is called an expansion of a hyperideal if $\delta : L(\Re) \longrightarrow L(\Re)$ is a function that satisfies the following properties:
i) $I \subseteq \delta(I)$, for all hyperideals I of $\Re$,
ii) If $I \subseteq J$, where I and J are hyperideals of $\Re$, then $\delta(I) \subseteq \delta(J)$.
iii) $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for all ideals I, J of $\Re$. Entire of the δ ideal expansions provides the property $\delta^2 = \delta$, which is $\delta(\delta(I)) = \delta(I)$, for all ideal I of $\Re$.

Definition 2.10. [21] An expansion function $\delta_{\Re}$ of hyperideals I of a hyperring $\Re$ is called $\delta_{\Re}$-primary if $a \circ b \in I$, then $a \in I$ or $b \in \delta_{\Re}(I)$, for all $a, b \in \Re$.

Remark 2.11. [21] Let δ_1 and δ_2 be two hyperideal expansions. Then the $\delta_1 \cap \delta_2$ is also a hyperideal expansion. Also δ_1 and δ_2 hyperideal expansions, $I \subseteq \delta_1(I)$ and $I \subseteq \delta_2(I)$, then $I \subseteq \delta_1(I) \cap \delta_2(I)$. Also, P and Q be any hyperideals of $\Re$ and $P \subseteq Q$. Hence, $\delta_1(P) \subseteq \delta_1(Q)$ and $\delta_2(P) \subseteq \delta_2(Q)$ it follows that $\delta_1(P) \cap \delta_2(P) \subseteq \delta_1(Q) \cap \delta_2(Q)$.

Remark 2.12. [21] A hyperideal expansion preserves intersection if $\delta_{\Re}(I \cap J) = \delta_{\Re}(I) \cap \delta_{\Re}(J)$ for any hyperideals I and J of $\Re$.

Remark 2.13. [21] Let $\Re$, S be any hyperrings, I be any hyperideal of S and σ be a good homomorphism such that $\sigma : \Re \to S$. If $\delta_{\Re}(\sigma^{-1}(I)) = \sigma^{-1}(\delta_{\Re}(I))$, then $\delta_{\Re}$ is called a global hyperideal expansion.

3 δ-primary subhypermodules on Krasner hyperrings

Throughout this section, $(\Re, \oplus, \circ)$ is a commutative Krasner hyperring with nonzero unit, M is an $\Re$-hypermodule with unitary and N is a proper subhypermodule of M. We denote the set of all subhypermodules of M by $L(M)$, the set of all hyperideals of $\Re$ by $L(\Re)$ and the set of all proper hyperideals of $\Re$, by $L^*(\Re)$. $\delta : L(\Re) \longrightarrow L(\Re)$ function is defined as a hyperideal expansion function that satisfies the following properties: $i)$ $I \subseteq \delta(I)$, for all hyperideals I of $\Re$, $ii)$ If $I \subseteq J$, where I and J are hyperideals of $\Re$, then $\delta(I) \subseteq \delta(J)$. Initially, we give the definition of δ-primary subhypermodule and present some examples.

Definition 3.1. Let N be a proper subhypermodule of an $\Re$-module M. N is said to be a δ-primary subhypermodule if $a \circ m \in N$, for each $a \in \Re$, $m \in M$, then $a \in \delta_{\Re}((N : M))$ or $m \in N$.

Example 3.2. Let $\Re$ be a commutative Krasner hyperring with a nonzero identity. Let us consider the following functions δ on $L(\Re)$, for any $I \in L(\Re)$:
(i) $\delta_0(I) = I$, i.e., δ is the identity function.
(ii) $\delta_1(I) = \sqrt{I}$, i.e., δ is the radical operation.
(iii) $\delta_{res}(I) = (I : J)$ for a fixed $J \in L(\Re)$.

(iv) $\delta_J(I) = I \oplus J$ for a fixed $J \in L(\Re)$.

All these functions are examples of expansion on $L(\Re)$.

Theorem 3.3. *Let $\Re$ be a hyperring and M be a hypermodule of $\Re$. Assume that N be a proper subhypermodule of M.*

(i) *N is a prime subhypermodule if and only if N is a δ_0-primary subhypermodule.*

(ii) *N is a primary subhypermodule if and only if N is a δ_1-primary subhypermodule.*

Proof. (i) Suppose that N is a prime subhypermodule of M and $a \circ m \in N$ for all $a \in \Re$, $m \in M$. So $m \in N$ or $a \in (N : M) = \delta_0(N : M)$. Thus N is a δ_0-primary subhypermodule. For the contrary, let N be a δ_0-primary subhypermodule of M, and $a \circ m \in N$ for all $a \in \Re$, $m \in M$. Then $a \in \delta_0(N : M)$ or $m \in N$. N is a prime hypersubmodule of M, since $\delta_0(N : M) = (N : M)$.

(ii) Suppose that N is a primary subhypermodule, and $a \circ m \in N$, for all $a \in \Re, m \in M$. Then $m \in N$ or $a^k \in (N : M)$, for some $k \in \mathbb{N}$. So it means $a \in \sqrt{(N : M)} = \delta_1(N : M)$. Then N is a δ_1-primary subhypermodule. Conversely, let N be a δ_1-primary subhypermodule of M, and $a \circ m \in N$ for all $a \in \Re$, $m \in M$. Then $a \in \delta_1(N : M)$ or $m \in N$ since N be a δ_1-primary subhypermodule. So $a \in \sqrt{(N : M)}$ because of $\delta_1(N : M) = \sqrt{(N : M)}$. Thus N is a primary subhypermodule of M. □

Let δ, γ be hyperideal expansions on $L(\Re)$. If $\delta_\Re(I) \subseteq \gamma_\Re(I)$, for each $I \in L(\Re)$, then we write $\delta \leq \gamma$.

Proposition 3.4. *Let M be an $\Re$-hypermodule and N be a proper subhypermodule of M and δ, γ be hyperideal expansions on $L(\Re)$.*

(i) *If $\delta \leq \gamma$, then every δ-primary subhypermodule is a γ-primary subhypermodule.*

(ii) *Every prime subhypermodule is a δ-primary subhypermodule.*

Proof. (i) Suppose that N is a δ-primary subhypermodule and $x \circ y \in N$ for every $x \in \Re$, $y \in M$. Then we have $x \in \delta_\Re(N : M)$ or $y \in N$. So $\delta_\Re(N : M) \subseteq \gamma_\Re(N : M)$, since $\delta \leq \gamma$. Finally we get that $x \in \gamma_\Re(N : M)$ or $y \in N$ which means N is γ-primary subhypermodule.

(ii) Let N be a prime subhypermodule of M and $x \circ y \in N$ and for any $x \in M$, $y \in \Re$. We know that $x \in N$ or $y \in (N : M)$ since N is prime subhypermodule of M. $(N : M) \subseteq \delta_{\Re}(N : M)$ since δ is a submodule expansion function. So $y \in \delta_{\Re}(N : M)$. Therefore $x \in N$ or $y \in \delta_{\Re}(N : M)$ which means N is a δ-primary subhypermodule of M. □

The converse of Proposition 3.4(ii) may not be true in general. If we take $\delta_1(I) = \sqrt{I}$, it is easy to see the subhypermodule $4\mathbb{Z}$ of the hypermodule $\mathbb{Z}$ is δ-primary subhypermodule but it is not prime subhypermodule.

Proposition 3.5. *Let δ be a hyperideal expansion and $\{L_i : i \in \Delta\}$ be a directed family of δ-primary subhypermodules of finitely generated $\Re$-hypermodule M. Then $\mathcal{L} = \bigcup_{i\in\Delta} L_i$ is a δ-primary subhypermodule.*

Proof. Let $\{L_i : i \in \Delta\}$ be a directed family of δ-primary subhypermodules of finitely generated hypermodule M. Assume that $a \circ m \in \mathcal{L}$, for all $a \in \Re$, $m \in M$. Indeed $a \circ m \in L_k$, for some $k \in \Delta$. We get either $a \in \delta_{\Re}(L_k : M)$ or $m \in L_k$, because of L_k is a δ-primary subhypermodule. If $m \in L_k$, then clearly we have $m \in \bigcup_{i\in\Delta} L_i = \mathcal{L}$. If $a \in \delta_{\Re}(L_k : M)$, then we have $a \in \delta(\bigcup_{i\in\Delta} L_i : M)$ since $L_k \subseteq \bigcup_{i\in\Delta} L_i = \mathcal{L}$. Hence $\mathcal{L}$ is a δ-primary subhypermodule of M. □

Considering the intersection of two δ-primary subhypermodules, it turns out that it is not δ-primary, since the intersection of two $\delta_{\Re}$-primary hyperideal is not $\delta_{\Re}$-primary.

Example 3.6. Let $2\mathbb{Z}$ and $3\mathbb{Z}$ be $\delta_{\Re}$-primary hyperideals. But $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not $\delta_{\Re}$-primary hyperideal of $\Re$, since while $3 \circ 4 \in 6\mathbb{Z}$, neither $3 \notin 6\mathbb{Z}$ nor $4 \notin \delta(6\mathbb{Z})$ (in other words neither $4 \notin 6\mathbb{Z}$ nor $3 \notin \delta(6\mathbb{Z})$).

Lemma 3.7. *Let N be a proper subhypermodule of $\Re$-hypermodule M. If N is δ-primary subhypermodule, then $(N : M)$ is a $\delta_{\Re}$-primary hyperideal of $\Re$.*

Proof. Assume that $a \circ m \in (N : M)$, for all $a, m \in \Re$ and $a \notin \delta_{\Re}(N : M) = \delta_{\Re}((N : M) : M))$, at the same time $a \notin (N : M) \subseteq \delta_{\Re}(N : M)$. Then $a \circ m \circ M \subseteq N$ and $a \circ M \nsubseteq N$. There is an element $t \in M$ such that $a \circ m \circ t \in N$ and $a \circ t \notin N$. So we get $m \in N \subseteq (N : M)$, since N is a δ-primary subhypermodule. Therefore $(N : M)$ is a $\delta_{\Re}$-primary hyperideal of $\Re$. □

Lemma 3.8. *Let N be a proper subhypermodule of $\Re$-hypermodule M. N is a δ-primary subhypermodule if and only if for each hyperideal J of $\Re$ and each subhypermodule L, $J \circ L \subseteq N$ implies that $J \subseteq \delta_{\Re}(N : M)$ or $L \subseteq N$.*

Proof. Suppose that N is a δ-primary subhypermodule. Let us assume $J \circ L \subseteq N$ and $a \in J$. Then there exists $t \in L \backslash N$ such that $a \circ t \in J \circ L \subseteq N$. We get $a \in \delta_{\Re}(N : M)$, since N is a δ-primary subhypermodule. Therefore $J \subseteq \delta_{\Re}(N : M)$. For the converse, $a' \circ t' \in N$ and $t' \notin N$. So $(a') \circ (t') \subseteq N$ and $(t') \nsubseteq N$. Then $a' \in (a') \subseteq \delta_{\Re}(N : M)$. Hereby N is a δ-primary subhypermodule of M. □

Lemma 3.9. *Let N be a proper subhypermodule of multiplication $\Re$-hypermodule M. N is a δ-primary subhypermodule if and only if $(N : M)$ is a $\delta_{\Re}$-primary hyperideal of $\Re$.*

Proof. Assume that N is δ-primary subhypermodule. Then, by Lemma 3.7, $(N : M)$ is a $\delta_{\Re}$-primary hyperideal of $\Re$. To the contrary, assume that $(N : M)$ is a $\delta_{\Re}$-primary hyperideal of $\Re$. Suppose that $J \circ K \subseteq N$ and $K \nsubseteq N$, where K is any subhypermodule of M and J is any hyperideal of $\Re$. There exists a hyperideal I such that $I \circ M = K$, since M is a multiplication $\Re$-hypermodule. Thus we can find $J \circ I \circ M \subseteq N$ and $J \circ I \subseteq (N : M)$. Also we have $I \circ M \nsubseteq N$, which means $I \nsubseteq (N : M)$. Therefore $J \subseteq \delta_{\Re}((N : M))$, since $(N : M)$ is a $\delta_{\Re}$-primary hyperideal. By Lemma 3.8, we find that N is a δ-primary subhypermodule of M. □

Theorem 3.10. *Let N be a proper subhypermodule of multiplication $\Re$-hypermodule M.*

(i) *If N is a δ-primary subhypermodule of M and J is a hyperideal such that $J \nsubseteq \delta_{\Re}((N : M))$, then $(N : J) = N$, where $(N : J) = \{m \in M : m \circ J \subseteq N\}$.*

(ii) *Let L be any δ-primary subhypermodule and Q be any subset of M. Then $(L : Q)$ is $\delta_{\Re}$-primary hyperideal, where $(L : Q) = \{r \in \Re : r \circ Q \subseteq L\}$.*

Proof. (i) Obviously $N \subseteq (N : J)$. On the other hand, $(N : J) \circ J \subseteq N$ and we assume that $J \nsubseteq \delta_{\Re}((N : M))$. $(N : J) \subseteq N$, since N is a δ-primary subhypermodule of M. Therefore $N = (N : J)$.

(ii) Suppose that $a \circ m \in (L : Q)$, for all $a, m \in \Re$, and $m \notin (L : Q)$. Then we can find $n \in Q$ such that $a \circ m \circ n \in L$ and $m \circ n \notin L$. From the hypothesis, L is a δ-primary subhypermodule. Then we get $a \in \delta_{\Re}((L : M))$. Hereby $(L : M) \subseteq (L : Q)$ means $\delta_{\Re}((L : M)) \subseteq \delta_{\Re}((L : Q))$. Therefore $a \in \delta_{\Re}((L : Q))$. □

Lemma 3.11. *Let L_i be δ-primary subhypermodules of M and $\delta_{\Re}((L_i : M))$ for $i = 1, 2, .., n$. If a hyperideal expansion $\delta_{\Re}$ preserves the intersection, then $\mathcal{L} = \bigcap_{i=1}^{n} L_i$ is δ-primary subhypermodule.*

Proof. Assume that $a \circ m \in \mathcal{L}$ and $m \notin \mathcal{L}$. We can find a $\Bbbk$ such that $a \circ m \in L_{\Bbbk}, m \notin L_{\Bbbk}$. Then $a \in \delta_{\Re}((L_{\Bbbk} : M))$, since $L_{\Bbbk}$ is δ-primary subhypermodule.

$$(\mathcal{L} : M) = (\bigcap_{i=1}^{n} L_i : M) = \bigcap_{i=1}^{n} (L_i : M)$$

and $\delta_{\Re}$ preserves intersection of hyperideal expansion. So

$$\delta_{\Re}((\mathcal{L} : M)) = \delta_{\Re}((\bigcap_{i=1}^{n} L_i : M)) = \bigcap_{i=1}^{n} \delta_{\Re}((L_i : M)).$$

Therefore $a \in \delta_{\Re}((\mathcal{L} : M))$. We get $\mathcal{L}$ is δ-primary subhypermodule of M. □

Now we give the notions of multiplication preserving, quotient preserving, and product in multiplication $\Re$-hypermodule.

Definition 3.12. Let M be a multiplication $\Re$-hypermodule and J be any hyperideal of $\Re$. If $\delta_{\Re}(J) \circ M = \delta(J \circ M)$, then an expansion δ is called multiplication preserving.

Example 3.13. Let $\mathbb{R}[x]$ denote the ring of polynomials over the real numbers $\mathbb{R}$ with variable x. We define a hyperideal J in $\mathbb{R}[x]$ as follows:

$$J = \langle x \rangle = \{f(x) \in \mathbb{R}[x] \mid \text{ the constant term of } f(x) \text{ is zero}\}.$$

Now, let $M = \mathbb{R}[x]$ itself be the multiplication $\mathbb{R}$-hypermodule, where the multiplication is given by polynomial multiplication. Define the expansion $\delta : \mathbb{R}[x] \to \mathbb{R}[x]$ as the identity map. That is, $\delta(f(x)) = f(x)$ for all $f(x) \in \mathbb{R}[x]$. If δ is multiplication preserving:

$$\delta(J) = \delta(\langle x \rangle) = \langle x \rangle.$$

$$J \circ M = \langle x \rangle \circ \mathbb{R}[x] = \{f(x)g(x) \mid f(x) \in \langle x \rangle, g(x) \in \mathbb{R}[x]\} = \langle x \rangle.$$
$$\delta_{\mathbb{R}}(J) \circ M = \delta_{\mathbb{R}}(\langle x \rangle) \circ \mathbb{R}[x] = \langle x \rangle \circ \mathbb{R}[x] = \langle x \rangle.$$

As $\delta_{\mathbb{R}}(J) \circ M = J \circ M$, the expansion δ is indeed multiplication preserving.

Definition 3.14. Let M be a multiplication $\Re$-hypermodule and N be a proper subhypermodule of M. An expansion δ is called quotient preserving if $\delta_{\Re}((N : M)) = (\delta(N) : M)$.

Example 3.15. Let M be the multiplication $\mathbb{R}$-hypermodule given by $M = \mathbb{R}[x]$, the ring of polynomials over the real numbers $\mathbb{R}$ with variable x. Consider the proper subhypermodule N defined as follows:

$$N = \{f(x) \in \mathbb{R}[x] \mid f(0) = 0\}.$$

That is, N consists of all polynomials whose constant term is zero. Let $\delta : \mathbb{R}[x] \to \mathbb{R}[x]$ be an expansion defined as follows:

$$\delta(f(x)) = f(x) + x.$$

If δ is quotient preserving:

$$(N : M) = \{r \in \mathbb{R} \mid rx \in N\} = \{0\}.$$
$$\delta_{\mathbb{R}}((N : M)) = \delta_{\mathbb{R}}(\{0\}) = \{0\}.$$
$$\delta(N) = \{f(x) + x \mid f(x) \in N\} = \{f(x) \mid f(0) = 0\} = N.$$
$$(\delta(N) : M) = (N : M) = \{0\}.$$

Therefore, we have $\delta_{\mathbb{R}}((N : M)) = (\delta(N) : M)$.

Definition 3.16. Let M be a multiplication $\Re$-hypermodule and N, L are subhypermodules of M, where $N = I \circ M$ and $L = J \circ M$, for some $I, J \in L(\Re)$. Denote the product of N and L by $N \circ L$ and define it by $I \circ J \circ M$. In other words, $q \circ t = Rq \circ Rt = I \circ J \circ M$, for $q, t \in M$.

Example 3.17. Consider the multiplication $\mathbb{R}$-hypermodule $M = \mathbb{R}^2$ with the usual component-wise scalar multiplication. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ denote the standard basis vectors of $\mathbb{R}^2$ and define two subhypermodules N and L of M as follows:

$$N = \langle e_1 \rangle = \{(x, 0) \mid x \in \mathbb{R}\},$$

$$L = \langle e_2 \rangle = \{(0, y) \mid y \in \mathbb{R}\}.$$

Let I and J represented by 2×2 matrices such that:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We compute the product of N and L, denoted by $N \circ L$:

$$N \circ L = I \circ J \circ M.$$

For any vector $v = (x, y) \in M$, we have:

$$I \circ J \circ v = I \circ (J(v)) = I \circ (0, y) = (0, 0).$$

Thus, $N \circ L$ consists solely of the zero vector.

The following theorem characterizes the behavior of subhypermodules in relation to N under the expansion δ and how to ensure that N is a primary subhypermodule with respect to δ in the hypermodule M.

Theorem 3.18. *Let M be a multiplication $\Re$-hypermodule, N be a proper subhypermodule of M and δ be an expansion which preserves quotient and multiplication. Then N is δ-primary subhypermodule if and only if for any two subhypermodules Q_1, Q_2 of M, if $Q_1 \circ Q_2 \subseteq N$, then $Q_1 \subseteq N$ or $Q_2 \subseteq \delta(N)$.*

Proof. Assume that N is δ-primary subhypermodule of M. Suppose $Q_1 \circ Q_2 \subseteq N$ and $Q_1 \nsubseteq N$, for any subhypermodules Q_1, Q_2 of M. We know that M is multiplication hypermodule, so there exist hyperideals I_1 and I_2 such that $Q_1 = I_1 \circ M$ and $Q_2 = I_2 \circ M$, respectively. We can find $I_1 \circ M \nsubseteq N$, then $I_1 \nsubseteq (N : M)$, since $Q_1 \nsubseteq N$. Therefore $Q_1 \circ Q_2 = I_1 \circ I_2 \circ M \subseteq N$ means $I_1 \circ I_2 \subseteq (N : M)$. Hence $I_2 \subseteq \delta_R((N : M))$. It follows $Q_2 = I_2 \circ M \subseteq \delta_R((N : M)) \circ M = (\delta(N) : M) \circ M = \delta(N)$ as δ is quotient and multiplication preserving. To the contrary, we assume N is a subhypermodule of M and $Q_1 \nsubseteq N$. There exist I_1 and I_2 hyperideals in $\Re$ such that $Q_1 = I_1 \circ M$ and $Q_2 = I_2 \circ M$ because of M is multiplication hypermodule. $Q_1 \circ Q_2 = I_1 \circ I_2 \circ M = (I_1 \circ M) \circ (I_2 \circ M) \subseteq N$. Then by the assumption $I_2 \circ M \subseteq \delta(N)$ which means $I_2 \subseteq (\delta(N) : M)$. Hereby $I_2 \subseteq \delta_{\Re}(N : M)$ as δ is quotient preserving. Therefore N is a δ-primary subhypermodule. □

Corollary 3.19. *Let M be a multiplication $\Re$-hypermodule, N be a proper subhypermodule of M and δ be an expansion which preserves quotient and multiplication. Then N is δ-primary subhypermodule if and only if $q \circ t \in N$, then $q \in N$ or $t \in \delta(N)$, for any $q, t \in M$.*

Proof. Suppose that N is δ-primary subhypermodule of M. By Theorem 3.18, we get the requested. Conversely, suppose $Q_1 \circ Q_2 \subseteq N$, $Q_1 \nsubseteq N$, for any subhypermodules Q_1, Q_2 of M. Let $t \in Q_2$. Then $\exists q \in Q_1 \setminus N$ such that $q \circ t \in Q_1 \circ Q_2 \subseteq N$. From the hypothesis $t \in \delta(N)$. Therefore $Q_2 \subseteq \delta(N)$ and N is δ-primary subhypermodule. □

Definition 3.20. Let $\Re$ be a hyperring. An element a of $\Re$ is said to be $\delta_{\Re}$-nilpotent, if $a \in \delta_{\Re}(\{0_{\Re}\})$.

Example 3.21. Consider the hyperring $\Re = \mathbb{R}[x]$, the ring of polynomials over the real numbers $\mathbb{R}$ with variable x. Define the expansion $\delta_{\Re} : \mathbb{R}[x] \to \mathbb{R}[x]$ as follows:

$$\delta_{\Re}(f(x)) = x \cdot f(x).$$

Now, let's find $\delta_{\Re}(\{0_{\Re}\})$, which is the image of the zero element of $\Re$ under $\delta_{\Re}$. Since $\{0_{\Re}\}$ consists only of the zero polynomial $0_{\Re}$, we have:

$$\delta_{\Re}(\{0_{\Re}\}) = \delta_{\Re}(0) = x \cdot 0 = 0.$$

Then an element $a \in \Re = \mathbb{R}[x]$ is $\delta_{\Re}$-nilpotent if $a \in \delta_{\Re}(\{0_{\Re}\}) = \{0\}$. An example of such an element would be any non-zero constant polynomial $a \in \mathbb{R}[x]$, because when multiplied by x, it yields the zero polynomial

$$x \cdot c = 0,$$

where $c \in \mathbb{R}$ is any non-zero constant. So, any non-zero constant polynomial in $\mathbb{R}[x]$ is $\delta_{\Re}$-nilpotent.

Now, we will show in the following theorem that the zero divisors of the quotient hypermodule $\Re/K$ are nilpotent under the action of $\delta_{\Re}$. It is a crucial property for understanding the structure and behavior of δ-primary subhypermodules within multiplication hypermodules.

Theorem 3.22. *Let M be a multiplication $\Re$-hypermodule, N be a proper subhypermodule of M and $\delta_\Re$ be a global expansion. N is δ-primary subhypermodule if and only if all zero divisors of $\Re/K$ are $\delta_\Re$-nilpotent for $K = (N : M)$.*

Proof. By Lemma 3.9, we need to prove K is $\delta_\Re$-primary hyperideal if and only if every zero divisor of $\Re/K$ is $\delta_\Re$-nilpotent. Suppose that $K = (N : M)$ is $\delta_\Re$-primary hyperideal of $\Re$. Let $\alpha = a \oplus K$ be any zero divisor of $\Re/K$. Then there exists a $\beta = b \oplus K \in \Re/K$ such that $\alpha \circ \beta = a \circ b \oplus K \subseteq 0_{\Re/K} = K$ with $b \notin K$. By the assumption K is $\delta_\Re$-primary hyperideal, then we have $a \in \delta(K)$ and $\alpha \subseteq \delta(K)/K$. $\sigma : \Re \to \Re/K$ is a natural homomorphism and δ is a global expansion, $\delta(K) = \delta(\sigma^{-1}(0_{\Re/K})) = \sigma^{-1}(\delta(\{0_{\Re/K}\}))$. Now on we find that $\delta(K)/K = \sigma(\delta(K) = \delta(\{0_{\Re/K}\})$ using the surjectivity of σ. So $\alpha \subseteq \delta(\{0_{\Re/K}\})$ and it is $\delta_\Re$-nilpotent. Conversely, suppose that every zero divisor of $\Re/K$ is $\delta_\Re$-nilpotent. Let $a \circ b \in K$, $b \notin K$, for any $a, b \in \Re$. Then $\alpha \circ \beta \subseteq 0_{\Re/K}$ with $\beta \neq 0_{\Re/K}$. So β is a zero divisor of $\Re/K$. Then $\alpha \subseteq \delta(\{0_{\Re/K}\} = \delta(K)/K$, since every zero divisor of $\Re/K$ is $\delta_\Re$-nilpotent. We can find a $q \in \delta(K)$ such that $a \ominus q \subseteq K \subseteq \delta(K)$. Therefore $a = (a \ominus q) \oplus q \subseteq \delta(K)$. Hence K is $\delta_\Re$-primary. □

Now, we will give the concepts of nilpotent subhypermodule, δ-nilpotent, zero divisor, and global homomorphism in a multiplication $\Re$-hypermodule.

Definition 3.23. Let M be a multiplication $\Re$-hypermodule, N be a subhypermodule of M. Then N is said to be nilpotent subhypermodule if $N^{\Bbbk} = 0$, for some $\Bbbk \in \mathbb{Z}^+$ and $q \in M$ is said to be nilpotent if $q^{\Bbbk} = 0$, $\exists \Bbbk \in \mathbb{Z}^+$.

Example 3.24. Let M be the set of 2×2 matrices over $\mathbb{R}$, and let N be the subhypermodule consisting of matrices of the form:

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

where $a \in \mathbb{R}$. For any matrix $A \in N$, raising $A^{\Bbbk}$ for any positive integer $\Bbbk$ will result in:

$$A^{\Bbbk} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^{\Bbbk} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $N^{\Bbbk} = \{0\}$ for any positive integer $\Bbbk$ and N is a nilpotent subhypermodule. Consider the matrix:

$$q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, $q^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which means q is nilpotent with $\Bbbk = 2$. Therefore, N is a nilpotent subhypermodule of M, and q is a nilpotent element of M.

Definition 3.25. Let M be a multiplication $\Re$-hypermodule. An element q of M is said to be δ-nilpotent, if $q \in \delta(\{0_M\})$.

Definition 3.26. Let M be a multiplication $\Re$-hypermodule. An element $a \neq 0_M$ is called a zero divisor in M when exists an element $m \neq 0_M$ in M such that $a \circ m = \Re \circ a \circ \Re \circ m = 0_M$.

Example 3.27. Consider the hypermodule $M = \mathbb{R}^2$ with the standard multiplication and addition of vectors. We define the scalar multiplication by $\lambda \circ (x, y) = (\lambda x, \lambda y)$ for $\lambda \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. Now, let's take the element $a = (1, 0) \neq (0, 0)$. And let $m = (0, 1) \neq (0, 0)$. Then we have:

$$\begin{aligned} a \circ m &= (1,0) \circ (0,1) = (1 \cdot 0, 0 \cdot 1) = (0,0) \\ \mathbb{R} \circ a \circ \mathbb{R} \circ m &= \mathbb{R} \circ (1,0) \circ \mathbb{R} \circ (0,1) \\ &= \lambda \circ (1,0) \circ \lambda \circ (0,1) = (\lambda \cdot 0, 0 \cdot \lambda) = (0,0) \end{aligned}$$

Thus, $a = (1, 0)$ is a zero divisor in $M = \mathbb{R}^2$.

Example 3.28. Let $\Re = \{0, 1, 2\}$ be a set with hyperoperation $\oplus$ and a binary operation $\odot$ as follows.

$\oplus$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0\}$

and

$\odot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(\Re, \oplus, \odot)$ is a hyperring. Let $M = \{0, a, b, c\}$ with the following hyperoperations.

$\boxplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$
b	$\{b\}$	$\{a,b\}$	$\{0\}$	$\{b,c\}$
c	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{0\}$

and

$\boxdot$	0	a	b	c
0	0	0	0	0
1	0	a	0	c
2	0	b	0	b

Then $(M, \boxplus, \boxdot)$ is a hypermodule over hyperring $(\Re, \oplus, \odot)$. Let subhypermodule $N = \{0, b\} \subseteq M$ and $I = \{0, 2\}$ be hyperideal of $\Re$. Since

$0 \boxdot 0 = 0$, $0 \boxdot a = 0$, $0 \boxdot b = 0$, $0 \boxdot c = 0$,

$2 \boxdot 0 = 0$, $2 \boxdot a = b$, $2 \boxdot b = 0$, $2 \boxdot c = b$,

then $\{0, b\} = N = I \circ M$ and M is a multiplication $\Re$-hypermodule. Let $x = 2 \neq 0$ and $m = b \neq 0$. As $x \boxdot m = 2 \boxdot b = 0$ and $\Re \odot x \odot \Re \boxdot m = 0$ such as

$\Re \odot x \odot \Re \boxdot m = 0 \odot x \odot 0 \boxdot m = 0 \odot 2 \odot 0 \boxdot b = 0 \odot 0 \boxdot b = 0 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 0 \odot x \odot 1 \boxdot m = 0 \odot 2 \odot 1 \boxdot b = 0 \odot 1 \boxdot b = 0 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 0 \odot x \odot 2 \boxdot m = 0 \odot 2 \odot 2 \boxdot b = 0 \odot 2 \boxdot b = 0 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 1 \odot x \odot 0 \boxdot m = 1 \odot 2 \odot 0 \boxdot b = 2 \odot 0 \boxdot b = 0 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 1 \odot x \odot 1 \boxdot m = 1 \odot 2 \odot 1 \boxdot b = 2 \odot 1 \boxdot b = 2 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 1 \odot x \odot 2 \boxdot m = 1 \odot 2 \odot 2 \boxdot b = 1 \odot 2 \boxdot b = 2 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 2 \odot x \odot 0 \boxdot m = 2 \odot 2 \odot 0 \boxdot b = 1 \odot 0 \boxdot b = 0 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 2 \odot x \odot 1 \boxdot m = 2 \odot 2 \odot 1 \boxdot b = 1 \odot 1 \boxdot b = 1 \boxdot b = 0$

$\Re \odot x \odot \Re \boxdot m = 2 \odot x \odot 2 \boxdot m = 2 \odot 2 \odot 2 \boxdot b = 1 \odot 2 \boxdot b = 2 \boxdot b = 0$

So, $x = 2$ is a zero divisor in M.

Definition 3.29. An expansion δ is called global homomorphism, where M and W are any $\Re$-hypermodules, $\sigma : M \to W$ is a hypermodule homomorphism and $\sigma^{-1}(\delta(N)) = \delta(\sigma^{-1}(N))$, for any N subhypermodule of W.

Example 3.30. Consider two $\mathbb{R}$-hypermodules $M = \mathbb{R}^2$ and $W = \mathbb{R}$, both with the standard scalar multiplication and addition of vectors. Now, define a map $\sigma : M \to W$ as $\sigma(x, y) = x$ for $(x, y) \in \mathbb{R}^2$. For all $(x_1, y_1), (x_2, y_2) \in M$,

$$\begin{aligned}\sigma((x_1, y_1) \circ (x_2, y_2)) &= \sigma(x_1 x_2, y_1 y_2) = x_1 x_2 \\ &= \sigma(x_1, y_1) \circ \sigma(x_2, y_2).\end{aligned}$$

Then, σ is a hypermodule homomorphism. Let $\delta : \mathcal{L}(M) \to \mathcal{L}(W)$ be defined by $\delta(N) = \{(x,y) \in M : \sigma(x,y) \in N\}$ for any subhypermodule $N \subseteq W$.

$$\begin{aligned}\sigma^{-1}(\delta(N)) &= \sigma^{-1}(\{(x,y) \in M : \sigma(x,y) \in N\}) \\ &= \{(x,y) \in M : \sigma(x,y) \in N\} \\ &= \{(x,y) \in M : x \in N\} \\ &= \{(x,y) \in M : \sigma(x,y) \in \sigma(\sigma^{-1}(N))\} \\ &= \delta(\sigma^{-1}(N)).\end{aligned}$$

Hence, δ is a global homomorphism.

The expansion δ plays a crucial role in characterizing the structure of N as a δ-primary subhypermodule. The following theorem shows how to ensure that the zero divisors of M/N are annihilated by some power of δ.

Theorem 3.31. *Let M be a multiplication $\Re$-hypermodule, N be a proper subhypermodule of M and δ be global, quotient and multiplication preserving expansion. Then N is δ-primary subhypermodule if and only if all zero divisors of M/N are δ-nilpotent.*

Proof. Suppose that N is a δ-primary subhypermodule of M. Let $\alpha = a \oplus N$ be any zero divisor of M/N. Then there exists a $\beta = b \oplus N \subseteq M/N$ such that $\alpha \circ \beta = a \circ b \oplus N \subseteq 0_{M/N} = N$ with $b \notin N$. By the assumption, we have $a \in \delta(N)$ and $\alpha \subseteq \delta(N)/N$. $\sigma : M \to M/N$ is a natural homomorphism and δ is global expansion, $\delta(N) = \delta(\sigma^{-1}(\{0_{M/N}\})) = \sigma^{-1}(\delta(\{0_{M/N}\}))$. Now on we find that $\delta(N)/N = \sigma(\delta(N) = \delta(\{0_{M/N}\})$ using the surjectivity of σ. So $\alpha \subseteq \delta(\{0_{M/N}\})$ and it is δ-nilpotent. Conversely, suppose that every zero divisor of M/N is δ-nilpotent. Let $a \circ b \in N$, $b \notin N$, for any $a, b \in M$. Then $\alpha \circ \beta \subseteq 0_{M/N}$ with $\beta \neq 0_{M/N}$. So β is a zero divisor of M/N. Then $\alpha \subseteq \delta(\{0_{M/N}\} = \delta(N)/N$, since every zero divisor of M/N is δ-nilpotent. We can find a $q \in \delta(N)$ such that $a \ominus q \subseteq N \subseteq \delta(N)$. Therefore $a = (a \ominus q) \oplus q \subseteq \delta(N)$. Hence N is δ-primary subhypermodule of M. □

Lemma 3.32. *Let M and W be two multiplication $\Re$-hypermodules, N be subhypermodule of W, $\sigma : M \to W$ be an onto hypermodule homomorphism and δ be global, quotient and multiplication preserving expansion. Then $\sigma^{-1}(N)$ is δ-primary subhypermodule of M, where N is δ-primary subhypermodule of W.*

Proof. Let $Q_1 \circ Q_2 \subseteq \sigma^{-1}(N)$, $Q_1 \nsubseteq \sigma^{-1}(N)$, for any subhypermodules Q_1, Q_2 of M. There exist hyperideals $I_1, I_2 \in \Re$ such that $Q_1 = I_1 \circ M, Q_2 = I_2 \circ M$, since M is multiplication hypermodule. Then we get $(I_1 \circ M) \circ (I_2 \circ M) = (I_1 \circ I_2) \circ M \subseteq \sigma^{-1}(N)$ and $I_1 \circ M \nsubseteq \sigma^{-1}(N)$. Since σ is onto, then $\sigma((I_1 \circ I_2) \circ M) \subseteq N$ and $\sigma(I_1 \circ M) \nsubseteq N$. So $(I_1 \circ I_2) \circ \sigma(M) \subseteq N$ and $I_1 \circ \sigma(M) \nsubseteq N$. It means $I_1 \circ I_2 \circ W \subseteq N$ and $I_1 \circ W \nsubseteq N$. By the hypothesis, $I_2 \circ W \subseteq \delta(N)$, so we have $\sigma(I_2 \circ M) \subseteq \delta(N)$. Therefore $I_2 \circ M \subseteq \sigma^{-1}(\delta(N)) = \delta(\sigma^{-1}(N))$ as δ is global homomorphism. Hence $Q_2 \subseteq \delta(\sigma^{-1}(N))$ and $\sigma^{-1}(N)$ is δ-primary subhypermodule of M. □

The following proposition indicates that the δ-primary subhypermodule property of N is preserved under the hypermodule homomorphism σ. This relationship between N and $\sigma(N)$ highlights the importance of understanding the structure of hypermodules.

Proposition 3.33. *Let M and W be two multiplication $\Re$-hypermodules, N be subhypermodule of M such that $N \supseteq \ker(\sigma)$ and $\sigma : M \to W$ be an onto hypermodule homomorphism. Let δ be global, quotient and multiplication preserving expansion. Then N is δ-primary subhypermodule of M if and only if $\sigma(N)$ is δ-primary subhypermodule of W.*

Proof. Suppose that N is a δ-primary subhypermodule of M. Let $Q_1 \circ Q_2 \subseteq \sigma(N)$, $Q_2 \nsubseteq \sigma(N)$, for any subhypermodules Q_1, Q_2 of W. There exist hyperideals $I_1, I_2 \in \Re$ such that $Q_1 = I_1 \circ W, Q_2 = I_2 \circ W$, since W is multiplication hypermodule. Then we get $(I_1 \circ W) \circ (I_2 \circ W) = (I_1 \circ I_2) \circ W \subseteq \sigma(N)$ and $I_2 \circ W \nsubseteq \sigma(N)$. Since σ is onto, $(I_1 \circ I_2) \circ M = (I_1 \circ M) \circ (I_2 \circ M) \subseteq N$ and $I_2 \circ M \nsubseteq N$. By the hypothesis, we have $I_1 \circ M \subseteq \delta(N)$. Therefore $I_1 \circ W \subseteq \sigma(\delta(N)) = \delta(\sigma(N))$ as δ is global homomorphism and σ is onto. Hence $Q_1 \subseteq \delta(\sigma(N))$ and $\sigma(N)$ is δ-primary subhypermodule of W. Conversely, suppose that $\sigma(N)$ is δ-primary subhypermodule of W. Since $N \supseteq \ker(\sigma)$, then $\sigma^{-1}(\sigma(N)) = N$ and we have N is δ-primary subhypermodule of M by Lemma 3.32. □

The following corollary demonstrates the expansion δ preserves the primary subhypermodule property when passing from M to M/N for the study of primary subhypermodules in quotient hypermodules.

Corollary 3.34. *Let M be a multiplication $\Re$-hypermodule, N, L be two subhypermodules of M such that $N \subseteq L$ and δ be a global, quotient and mul-*

tiplication preserving expansion. Then L/N is δ-primary subhypermodule of M/N if and only if L is δ-primary subhypermodule of M.

Proof. It is obvious by Lemma 3.32 and Proposition 3.33. □

4 Conclusion

In this paper, characterizations of the δ-primary subhypermodule were provided using the expansion function δ. We introduced the δ-primary subhypermodule, and several characterizations to classify them were obtained. Then, we investigated whether the union of the collection of δ-primary subhypermodules preserves the algebraic structure. Besides, we examined the images and inverse images of δ-primary subhypermodules under homomorphism. Also, we provided some characterizations for multiplication hypermodules with some special conditions. The results presented in this paper contribute to a better understanding of the structure of hypermodules with respect to the expansion function δ. These characterizations are essential for further research in this area and provide a foundation for future studies on δ-primary subhypermodules. We investigated many behaviors of δ-primary subhypermodules in particular cases. Our findings shed light on the intricate relationships between hypermodules and δ-primary subhypermodules, paving the way for deeper exploration into their properties. This analysis opens up new avenues for studying the behavior of subhypermodules under the expansion function δ in various contexts. In future work, one can generalize this study. Future research could explore the application of these characterizations in other areas of mathematics or investigate the properties of δ-primary subhypermodules in different algebraic structures. We suggest open problems to researchers based on our findings.
(1) To consider 1-absorbing δ-primary subhypermodules on Krasner hyperrings,
(2) To describe 2-absorbing δ-primary subhypermodules on Krasner hyperrings,
(3) To think ϕ-δ-primary subhypermodules on Krasner hyperrings,
(4) To think weakly δ-primary subhypermodules on Krasner hyperrings,
(5) To think S-δ-primary subhypermodules on Krasner hyperrings.

Compliance with Ethical Standards

Conflict of Interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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Elif Kaya *Department of Mathematics and Science Education, Istanbul Sabahattin Zaim University, Istanbul, Turkiye.*
Email: elif.kaya@izu.edu.tr

Melis Bolat *Department of Computer Engineering, Istanbul Gelisim University, Istanbul, Turkiye.*
Email: mbolat@gelisim.edu.tr

Bayram Ali Ersoy *Department of Mathematics, Yildiz Technical University, Istanbul, Turkiye.*
Email: ersoya@yildiz.edu.tr

Serkan Onar *Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkiye.*
Email: serkan10ar@gmail.com

Kostaq Hila *Department of Mathematical Engineering, Polytechnic University of Tirana, Albania.*
Email: kostaq_hila@yahoo.com

Bijan Davvaz *Department of Mathematics, Yazd University, Yazd, Iran.*
Email: davvaz@yazd.ac.ir