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Research Article

r-Hyperideals and Generalizations of r-Hyperideals in Krasner Hyperrings

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This paper deals with Krasner hyperrings as an important class of algebraic hyperstructures. We investigate some properties of r-hyperideals in commutative Krasner hyperrings. Some properties of pr-hyperideals are also studied. The relation between prime hyperideals and r-hyperideals is investigated. We show that the image and the inverse image of an r-hyperideal are also an r-hyperideal. We also introduce a generalization of r-hyperideals, and we prove some properties of them.

1. Introduction

Prime ideals and primary ideals play a significant role in commutative ring theory. Numerous generalizations of prime ideals have been studied by different researchers. The concepts of ϕ-prime and ϕ-primary ideals are examined in [1, 2].

French mathematician Marty [3] initiated the study of hyperstructures, as an expansion of classical algebraic structures, in special, of hypergroups, at the 8th Congress of Scandinavian Mathematicians in 1934. Subsequently, numerous articles and various books have been published about this issue. This theory has been studied by many researchers [4–17].

Hyperrings generalize rings, and they arise naturally in several subfields in algebra, including quadratic form theory, number theory, orderings and ordered algebraic structures, tropical geometry, and multiplicative subgroups of fields. Hyperrings $(\mathfrak{R}, +, \cdot)$ are of different types. If $+$ is a binary operation and the multiplication $\cdot$ is a hyperoperation, then the hyperring is called multiplicative hyperring [18]. Krasner hyperrings are an important class of algebraic hyperstructures that generalize rings further to allow multiple output values for the addition operation [12]. In fact, Krasner introduced the concepts of hyperrings and hyperfields in 1956 [19]. In 2006, the same notions called "multirings" and "multifields" were introduced independently by Marshall [20], with the only difference between hyperrings and multirings to be that hyperrings have the strong distributive property whereas multirings have the weak distributive property. Later on, after Krasner, Stratigopoulos [21] and Mittas [13, 14, 22] have initiated the general study of these algebraic hyperstructures. There are several well-known authors that have made an important contribution to the study of hyperrings and hyperfields later on and nowadays such as Massouros [23–25], Nakassis [26], Vougiouklis [27], Spartalis [28, 29], G. Pinotsis, Y. Kemprasit, M. Stefanescu, V. Leoreanu, R. Ameri, I. Cristea, and many others. The study of Krasner hyperrings has been recently one of the mainstream objective of the researchers [9, 30–32].

In 2015, Mohamadian [33] investigated some properties of r-ideals in commutative rings. While the research of Erbay et al. [34] focuses on r-ideals in commutative semigroups, Koc and Tekir in [35] considered a generalization of this notion to modules. In [36], Ugurlu studied generalizations of r-ideals. The concept of r-hyperideals in commutative hyperrings was briefly mentioned in [37], as a generalization of r-ideals in commutative rings.

In this paper, our aim is to extend the notion of r-ideals to r-hyperideals in Krasner hyperrings and generalize them. Some properties of r-hyperideals in the commutative Krasner hyperrings are obtained. Some properties of pr-hyperideals are also studied. The relation between prime hyperideals and r-hyperideals is investigated. It is shown that if N is a minimal prime hyperideal, then it is also an r-hyperideal. Furthermore, maximal hyperideals and r-hyperideals are compared. We show that the image and the inverse image of an r-hyperideal are also an r-hyperideal. Moreover, a generalization of r-hyperideals is introduced. ϕ-prime and ϕ-primary hyperideals, $\phi - r$-hyperideals, $\phi - pr$-hyperideals, $\phi-$ pure hyperideals, and $\phi-$ von Neumann regular hyperideals in $\mathfrak{R}$ are introduced and studied. Our study serves as a continuation of the study in more depth of the results of published papers reflected in the references on Krasner hyperrings, introducing and investigating these new classes of hyperideals.

Let $\mathfrak{R}$ be a commutative Krasner hyperring, N be a proper hyperideal of $\mathfrak{R}$, and $\phi\colon I\,d(\mathfrak{R}) \longrightarrow I\,d(\mathfrak{R}) \cup \{\varnothing\}$ be a function. $I\,d(\mathfrak{R})$ denotes the hyperideals of $\mathfrak{R}$. N is called $\phi - r$-hyperideal (resp., $\phi - pr$-hyperideal) if $a \cdot b \in N - \phi(N)$ with $ann(a) = 0$ implies that $b \in N$ (resp., $b^n \in N$), for $a, b \in \mathfrak{R}$. Some properties of them are provided. As a result, we obtain that the union of directed collection of ascending chain $\phi - r$-hyperideals of $\mathfrak{R}$ is also $\phi - r$-hyperideal of $\mathfrak{R}$, when ϕ preserves the order.

2. Preliminaries

For convenience, let us first give the definitions of $\phi-$ prime ideal, $\phi-$ primary ideal, and r-ideal in a commutative ring.

Definition 1 (see [38]). Let $\mathfrak{R}$ be a commutative ring. A reduction of ideals is a function ϕ that leads any ideal N of $\mathfrak{R}$ to other ideal $\phi(N)$ of $\mathfrak{R}$ such that the following statements hold:

(i) For all ideals N of $\mathfrak{R}$, $\phi(N) \subseteq N$

(ii) If $N \subseteq M$ where N and M are ideals of $\mathfrak{R}$, then $\phi(N) \subseteq \phi(M)$

Definition 2 Let $\mathfrak{R}$ be a commutative ring. Let N be an ideal of $\mathfrak{R}$. For $a, b \in \mathfrak{R}$,

(i) N is said to be a $\phi-$ prime ideal, if $ab \in N - \phi(N)$, then $a \in N$ or $b \in N$ [1]

(ii) N is said to be a $\phi-$ primary ideal, if $ab \in N - \phi(N)$, then $a \in N$ or $b \in \sqrt{N}$ [2]

Definition 3 (see [33]). Let $\mathfrak{R}$ be a commutative ring. A proper ideal N of $\mathfrak{R}$ is called an r-ideal (resp., pr-ideal), if $ab \in N$ with $ann(a) = (0)$ implies that $b \in N$ (resp., $b^n \in N$, for any $n \in \mathbb{N}$), for each $a, b \in \mathfrak{R}$.

Definition 4 (see [36]). Let $\mathfrak{R}$ be a commutative ring. A proper ideal N of $\mathfrak{R}$ is called a $\phi - r$-ideal, if $ab \in N - \phi(N)$ with $ann(a) = (0)$ implies that $b \in N$ for $a, b \in \mathfrak{R}$. Furthermore, N is called a $\phi - pr$-ideal, if $ab \in N - \phi(N)$ with $ann(a) = (0)$ implies that $b^n \in N\,(n \in \mathbb{N})$ for $a, b \in \mathfrak{R}$.

In the following, we recall some notions regarding hyperstructure.

Let G be a nonempty set and $P^*(G)$ denotes the family of all nonempty subsets of G. A mapping $\circ\colon G \times G \longrightarrow P^*(G)$ is called a binary hyperoperation on G. The couple $(G, ^\circ)$ is called a hypergroupoid [3]. In the above definition, if A and B are two nonempty subsets of G and $x \in G$, then we define

$$\begin{aligned} A \circ B &= \bigcup_{a \in A, b \in B} a \circ b, \\ A \circ x &= A \circ \{x\}, \text{ and} \\ x \circ B &= \{x\} \circ B. \end{aligned} \tag{1}$$

A hypergroupoid $(H, \circ)$ is said to be a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v. \tag{2}$$

When $(G, \circ)$ is a hypergroupoid, if there is $e \in G$ such that $x \in (e^\circ x) \cap (x^\circ e)$, for every $x \in G$, then e is called the identity element.

Semihypergroup $(G, \circ)$ is said to be a hypergroup if $x \circ G = G \circ x = G$, $\forall x \in G$ [3]. If $(G, \circ)$ is a hypergroup, $\varnothing \neq H$ is a subset of G and $x \circ H = H \circ x = H$, $\forall x \in H$, then $(H, \circ)$ is called a subhypergroup of $(G, \circ)$.

Let $(G, \circ)$ be a hypergroup. If $x \circ y = y \circ x$, $\forall x, y \in G$, then $(G, \circ)$ is called a commutative hypergroup.

Definition 5 (see [14]). A nonempty set $\mathfrak{R}$ along with the hyperoperation $+$ is called a canonical hypergroup if the following axioms hold:

(i) $x + (y + z) = (x + y) + z$, for $x, y, z \in \mathfrak{R}$

(ii) $x + y = y + x$, for $x, y \in \mathfrak{R}$

(iii) There exists $0 \in \mathfrak{R}$ such that $x + 0 = \{x\}$, for any $x \in \mathfrak{R}$

(iv) For any $x \in \mathfrak{R}$, there exists a unique element $x' \in \mathfrak{R}$, such that $0 \in x + x'$ (x' is called the opposite of x and it is denoted by $-x$)

(v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$; that is, $(\mathfrak{R}, +)$ is reversible

Definition 6 (see [12]). $(\mathfrak{R}, +, \cdot)$ is called a Krasner hyperring if it satisfies the following conditions:

(1) $(\mathfrak{R}, +)$ is a canonical hypergroup

(2) $(\mathfrak{R}, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = 0$, for all $x \in \mathfrak{R}$

(3) $(y+z)\cdot x=(y\cdot x)+(z\cdot x)$ and $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$, for all $x,y,z\in\mathfrak{R}$

Example 1 (for more details, see [39, 40]). Let $H=[0,1]$. Then, $(H,\oplus,\cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication, and the hyperaddition $\oplus$ is defined by

$$x\oplus y=\begin{cases}\{\max\{x,y\}\}, & \text{if } x\neq y,\\ [0,x], & \text{if } x=y.\end{cases}\tag{3}$$

Lemma 1 (see [41]). *A nonempty subset A of a Krasner hyperring $\mathfrak{R}$ is a left (resp. $r\cdot x\in A$ (resp right) hyperideal if and only if*

(i) $x-y\subseteq A$, for $x,y\in A$

(ii) $r\cdot x\in A$ (resp $x\cdot r\in A$), for $x\in A, r\in\mathfrak{R}$

Definition 7 (see [42]). Let N be a hyperideal of a hyperring $\mathfrak{R}$. Then, $\sqrt{N}=\{r\in\mathfrak{R}\colon r^n\in N$, for some $n\in\mathbb{N}\}$.

Let $\mathfrak{R}$ be a hyperring. For $a\in\mathfrak{R}$, we define $ann(a)=\{r\in\mathfrak{R}\colon ra=0\}$. If $ann(a)=0$ (resp., $ann(a)\neq 0$), a is said to be regular (resp., zero divisor). We use the notion $r(\mathfrak{R})$ (resp, $z\ d(\mathfrak{R})$) to denote the set of all regular elements (resp., zero divisors) [43]. If N is a hyperideal of $\mathfrak{R}$ and A a subset of $\mathfrak{R}$, then we denote $(N\colon A)=\{x\in\mathfrak{R}, x\cdot A\subseteq N\}$. It is clear that $(0\colon A)=ann(A)$.

Definition 8 (see [37]). A proper hyperideal N of a commutative Krasner hyperring $\mathfrak{R}$ is called an r-hyperideal (pr-hyperideal), if $a\cdot b\in N$ and $ann(a)=0$ implies that $b\in N$ ($b^n\in N$, for some $n\in\mathbb{N}$), for any $a,b\in\mathfrak{R}$.

$\mathrm{Min}(N)$ is the set of all minimal prime hyperideals which contain N. $\mathrm{Min}((0))$ is stated by $\mathrm{Min}(\mathfrak{R})$.

Definition 9 (cf. [44]). A hyperring $\mathfrak{R}$ satisfies the following:

(i) Property A: if any finitely generated hyperideal, $N\subseteq z\ d(\mathfrak{R})$ has nonzero annihilator

(ii) Annihilator condition: if for any finitely generated hyperideal N of $\mathfrak{R}$, there exists an element $a\in\mathfrak{R}$ such that $ann(N)=ann(a)$

(iii) Strong annihilator condition (briefly s.a.c.): if for any finitely generated hyperideal N of $\mathfrak{R}$, there exists an element $a\in N$ such that $ann(N)=ann(a)$

N is a z^0-hyperideal if $a\in N$, $b\in\mathfrak{R}$ and $ann(a)=ann(b)$ imply that $b\in N$.

$soc(\mathfrak{R})$ denotes the sum of all minimal hyperideals of $\mathfrak{R}$. The socle of a reduced hyperring $\mathfrak{R}$ is called $soc(\mathfrak{R})=\oplus_{i\in A}e_i\mathfrak{R}$, where $\{e_i\colon i\in A\}$ is the set of idempotents of $\mathfrak{R}$ [45].

Let A be a subset of $\mathfrak{R}$ and $a\in A$ such that $a\in a\mathfrak{R}a$. Then, a is called von Neumann regular element. Therefore, if all of the elements are von Neumann regular, A or $\mathfrak{R}$ is called von Neumann regular [46].

A two-sided hyperideal N in a semihypergroup $\mathfrak{R}$ is called right pure hyperideal if, for each $a\in N$, there exists $b\in N$ such that $a\in ab$ [47]. Similarly, a hyperideal is called pure hyperideal in a Krasner hyperring, if for each $a\in N$, there exists $b\in N$ such that $a=ab$.

A hyperring $\mathfrak{R}$ is called a reduced hyperring if there are no nilpotent elements in $\mathfrak{R}$. If $x^n=0$ for $x\in\mathfrak{R}$, $n\in\mathbb{N}$, then $x=0$ [48].

Let $f\colon\mathfrak{R}\longrightarrow S$ be a homomorphism, N be a hyperideal of $\mathfrak{R}$, and M be a hyperideal of S. Then, the hyperideal $\langle f(N)\rangle$ is said to be the extension of N, and it is denoted by N^e. The hyperideal $f^{-1}(M)$ is called contraction of M and denoted by M^c [26]. The mapping $\varphi\colon\mathfrak{R}\longrightarrow S^{-1}\mathfrak{R}$ given by $a=a/1$ is a homomorphism. If N is a hyperideal of $\mathfrak{R}$, then $\varphi(N)=S^{-1}N=\{i/s\in S^{-1}\mathfrak{R}\colon i\in N, s\in S\}$ is also a hyperideal of $S^{-1}\mathfrak{R}$ [10]. If $S=r(\mathfrak{R})$, then the hyperring $S^{-1}\mathfrak{R}$ is called quotient hyperring which is denoted by $Q(\mathfrak{R})$.

3. r-Hyperideals

In this section, we examine some properties of r-hyperideals in commutative hyperrings. We compare r-hyperideals with prime and maximal hyperideals. Throughout the section, $(\mathfrak{R},+,\cdot)$ is a commutative Krasner hyperring.

Initially, let us give some examples for better understanding.

Example 2. Clearly, $\mathbb{Z}_6$ is a Krasner hyperring with the usual addition and multiplication, and $\{\hat{0},\hat{2},\hat{4}\}=N$ is a proper hyperideal.

$ann(\hat{5})=0$ and $\hat{0}\cdot\hat{5}=\hat{0}\in N$ implies that $\hat{0}\in N$
$ann(\hat{5})=0$ and $\hat{2}\cdot\hat{5}=\hat{4}\in N$ implies that $\hat{2}\in N$
$ann(\hat{5})=0$ and $\hat{4}\cdot\hat{5}=\hat{2}\in N$ implies that $\hat{4}\in N$

Then N is an r-hyperideal of $\mathbb{Z}_6$.

Example 3 (see [49]). Let $H=\{0,1,a\}$ and $B=\{0,a\}$. Then, $(H,+,\cdot)$ is a Krasner hyperring with the hyperaddition and multiplication defined by

$+$	0	1	a
0	0	1	a
1	1	H	1
a	a	1	B

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

Clearly, B is an r-hyperideal of H.

Example 4. Let $R=\{0,a,b,c\}$ be a set with the hyperaddition $\oplus$ and multiplication $\odot$ defined as follows:

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{0,b\}$	$\{a,c\}$	b
b	b	$\{a,c\}$	$\{0,b\}$	a
c	c	b	a	0

Then, $(R,\oplus,\odot)$ is a Krasner hyperring [48]. It can be easily seen that $I_1=\{0,b\}, I_2=\{0,c\}, I_3=\{0,b,c\}$ are r-hyperideals.

Our main results regarding r-hyperideals are the followings.

Theorem 1. *Let $\mathfrak{R}$ be a hyperring and N be a hyperideal of $\mathfrak{R}$. Then, the following statements are equivalent:*

(a) N is an r-hyperideal

(b) $(a\cdot\mathfrak{R})\cap N = a\cdot N$, for any $a\in r(\mathfrak{R})$

(c) $N=(N\colon a)$, for any $a\in r(\mathfrak{R})/N$

(d) $N=M^c$, where M is a hyperideal of $Q(\mathfrak{R})$

Proof. ($a\Rightarrow b$) Let N be an r-hyperideal and a be a regular element. Suppose that $x\in\ (a\cdot\mathfrak{R})\cap N$. Then, $x\in a\cdot\mathfrak{R}$ and $x\in N$. Thus, $x=a\cdot a\prime$, for $a'\in\mathfrak{R}$. Since $x=a\cdot a\prime\in N$, $ann(a)=0$ and N is an r-hyperideal, $a'\in N$. Hence, $a\cdot a'\in a\cdot N$ and $(a\cdot\mathfrak{R})\cap N\subseteq a\cdot N$.

Therefore, for every $a\in\mathfrak{R}$, $a\cdot N\subseteq(a\cdot\mathfrak{R})\cap N$, $(a\cdot\mathfrak{R})\cap N=a\cdot N$.

($b\Rightarrow c$) We know that $N\subseteq(N\colon a)$, for every $a\in\mathfrak{R}$. Let a be regular, $a\notin N$ and $x\in(N\colon a)$. Hence, $ann(a)=0$ and $x\cdot a\in N$. From (b), $a\cdot x\in(a\cdot\mathfrak{R})\cap N$ and $a\cdot x\in a\cdot N$. This implies that $a\cdot x=a\cdot y$, for $y\in N$. Since $ann(a)=0$, then $x=y\in N$. Hence, $x\in N$. Thus, $(N\colon a)\subseteq N$.

($c\Rightarrow d$) Let S be the set of regular elements and $\varphi\colon\mathfrak{R}\longrightarrow Q(\mathfrak{R})$ be a natural homomorphism. We know that $N\subseteq\varphi^{-1}(M)=M^c$, for M a hyperideal of $Q(\mathfrak{R})$. Suppose $x\in\varphi^{-1}(M)$. Since $\varphi(x)=x/s\in S^{-1}\mathfrak{R}$, then $s\cdot x\in N$, for $s\in S$. From (c), $x\in(N\colon s)=N$.

($d\Rightarrow a$) Let $a\cdot x\in N$ and $ann(a)=0$. We have $(a\cdot x)/1=a/1\cdot x/1\in M$, and since a is regular, then there exists $1/a$ which is the inverse of $a/1$ in S. Thus, $(a/1)\cdot(x/1)\cdot(1/a)\in M$ and $(x/1)\in M$. Hence, $x\in N$ and so N is an r-hyperideal. □

Corollary 1. *The following statements hold:*

(a) The zero hyperideal is an r-hyperideal

(b) The intersection of r-hyperideals is an r-hyperideal

(c) When N is an r-hyperideal, $N\subseteq z\ d(\mathfrak{R})$

(d) Every r-hyperideal is a pr-hyperideal

(e) A prime hyperideal is an r-hyperideal if and only if it consists all of zerodivisors. As a result, every minimal prime hyperideal is an r-hyperideal

(f) Let N be an r-hyperideal, $S\subseteq\mathfrak{R}$ and $S\nsubseteq N$. Then, $(N\colon S)$ is an r-hyperideal. Particularly, $ann(x)$ and $ann(S)$ are always r-hyperideals.

(g) Every minimal hyperideal of a reduced hyperring is an r-hyperideal

(h) Every pure hyperideal and every von Neumann regular hyperideal are r-hyperideals

(i) Suppose that $\mathfrak{R}$ satisfies the s.a.c. and N is a hyperideal of $\mathfrak{R}$.N is an r-hyperideal if and only if for every hyperideal J and K of $\mathfrak{R}$ such that J is finitely generated, whenever $J\cdot K\subseteq N$ and $ann(J)=0$, then $K\subseteq N$

(j) The sum of two r-hyperideals may not be an r-hyperideal

Proof

(b) Suppose that $I_1,I_2,\ldots,I_n$ are r-hyperideals of $\mathfrak{R}$. Let $r\cdot x\in\cap_{i=1}^{n}I_i$ and $ann(r)=0$. Then, $r\cdot x\in I_i$. Since every I_i is r-hyperideal, then we have $x\in I_i$. Hence, $x\in\cap_{i=1}^{n}I_i$., which means the intersection of r-hyperideals is also r-hyperideal.

(c) Let N be an r-hyperideal of $\mathfrak{R}$ and $N\nsubseteq z\ d(\mathfrak{R})$. Then, there exists a regular element r in N. Now, let e be identity element of $\mathfrak{R}$. In this way, $e\cdot r\in N$. Since N is an r-hyperideal, then $e\in N$. This is a contradiction to N's being proper hyperideal.

(d) If a prime hyperideal N is an r-hyperideal, then it consists all of zerodivisors from (c). For the converse, suppose that N is prime and $N\subseteq z\ d(\mathfrak{R})$. Let $r\cdot x\in N$ and $ann(r)=0$. Since N is prime, then $r\in N$ or $x\in N$. Since we assume $N\subseteq z\ d(\mathfrak{R})$, then there is no regular element in N and so $r\notin N$. Then, we get $x\in N$. We conclude that N is an r-hyperideal.

(i) Assume that N is an r-hyperideal, J is finitely generated hyperideal, and K is a hyperideal of $\mathfrak{R}$ such that $J\cdot K\subseteq N$ and $ann(J)=0$. Then, there exists a $b\in J$ such that $ann(J)=ann(b)$ and so $ann(b)=0$. For $k\in K$, we have $b\cdot k\in J\cdot K\subseteq N$. Since N is an r-hyperideal, then $k\in N$. That's why $K\subseteq N$. Conversely, assume that for every hyperideal J and K of $\mathfrak{R}$ such that J is finitely generated, whenever $J\cdot K\subseteq N$ and $ann(J)=0$, then $K\subseteq N$. Let $a\cdot\mathfrak{R}=J$ and $b\cdot\mathfrak{R}=K$. Then, $J\cdot K=a\cdot b\cdot\mathfrak{R}\subseteq N$ and $ann(a)=0$. Therefore, $b\cdot\mathfrak{R}\subseteq N$ because of $K\subseteq N$. Thus, $b\in N$, and we get the desired result.

(j) To prove that the sum of two r-hyperideals may not be an r-hyperideal, we give the following example:

Example 5. Let $(G,\odot)$ be a group and $H=G\cup\{0,u,v\}$, where 0 is an absorbing element under multiplication and u,v are distinct orthogonal idempotents with

$u\odot v=v\odot u=0; u\odot u=u$

$v\odot v=v; a\odot 0=0\odot a=0$; for all $a\in H$

$u\odot g=g\odot u=u; v\odot g=g\odot v=v$; for all $g\in G$

If we define the hyperaddition $\oplus$ on H as follows:

$a\oplus 0=0\oplus a=\{a\}; a\oplus a=\{0,a\}$, for all $a\in H$

$a\oplus b=b\oplus a=H/\{0,a,b\}$, for all $a,b\in H/\{0\}$ and $a\neq b$

Then, $(H,\oplus,\odot)$ is a Krasner hyperring [41]. Consider $I=\{0,u\}$ and $J=\{0,v\}$. It can be easily seen that I,J are r-hyperideals, while $I\oplus J=H$ is not an r-hyperideal.

We omit the rest of proof since it is obvious.

Remark 1. Let K and L be hyperideals of $Q(\mathfrak{R})$. We know that $K^c\cdot L^c\subseteq(K\cdot L)^c$ and $K^c+L^c\subseteq(K+L)^c$. If N and M are

r-hyperideals of $\mathfrak{R}$, then by Theorem 1(d),$N = K^c$ and $M = L^c$, for some hyperideals K and L in $Q(\mathfrak{R})$. It follows that

(a) $N \cdot M$ is an r-hyperideal of $\mathfrak{R}$ if and only if $(K \cdot L)^c \subseteq K^c \cdot L^c$ (essentially $(K \cdot L)^c = K^c \cdot L^c$)

(b) $N + M$ is an r-hyperideal of $\mathfrak{R}$ if and only if $(K + L)^c \subseteq K^c + L^c$ (essentially $(K + L)^c = K^c + L^c$)

Lemma 2. *Let $\mathfrak{R}$ be a hyperring and N be a hyperideal. Then, the following statements hold:*

(a) N is an r-hyperideal if and only if for any J, K hyperideals of $\mathfrak{R}$ with $J \cap r(\mathfrak{R}) \neq \varnothing$ and $J \cdot K \subseteq N$, then $K \subseteq N$

(b) Assume that $N \subseteq z\ d(\mathfrak{R})$ is not an r-hyperideal. There exist hyperideals J and K such that $J \cap r(\mathfrak{R}) \neq \varnothing$, $N \subsetneq J$, K, and $J \cdot K \subseteq N$

Proof

(a) Let N be an r-hyperideal, J and K be hyperideals of $\mathfrak{R}$ such that $J \cap r(\mathfrak{R}) \neq \varnothing$ and $J \cdot K \subseteq N$. Let $a \in J \cap r(\mathfrak{R})$ and $b \in K$. Since J and K are hyperideals, then we can take $a \cdot \mathfrak{R} \subseteq J$ and $b \cdot \mathfrak{R} \subseteq K$. By our assumption, $a \cdot b \cdot \mathfrak{R} \subseteq N$. Since $J \cap r(\mathfrak{R}) \neq \varnothing$, then let we take $ann(a) = 0$. Then N is an r-hyperideal and so $b \cdot \mathfrak{R} \subseteq N$. Thus, $K \subseteq N$.

Conversely, assume that $J \cap r(\mathfrak{R}) \neq \varnothing$ and $J \cdot K \subseteq N, K \subseteq N$. Let $a \cdot \mathfrak{R} \subseteq J$ and $b \cdot \mathfrak{R} \subseteq K$. Then, $a \cdot b \cdot \mathfrak{R} \subseteq N$. Let $a \in J \cap r(\mathfrak{R})$. So $ann(a) = 0$. At the same time, since $K \subseteq N$, then $b \cdot \mathfrak{R} \subseteq N$.

(b) Assume that $N \subseteq z\ d(\mathfrak{R})$ is not an r-hyperideal. Then, there exist $r \in r(\mathfrak{R})$ and $x \notin N$ such that $r \cdot x \in N$. We have $ann(r) = 0$. Let $J = (N{:}\ x)$ and $K = (N{:}\ J)$. It follows $r \in J$. Since $ann(r) = 0$, then $J \cap r(\mathfrak{R}) \neq \varnothing$ and $r \notin N$. Since $N \subseteq z\ d(\mathfrak{R})$, then $ann(x) \neq 0$. Therefore, $x \in K$. Thus, there exist $b, c \in \mathfrak{R}$ such that $b \cdot x \in N$ and $c \cdot J \subseteq N$. Hence, $b \cdot x \cdot c \cdot J \subseteq N$. Then, $x \cdot J \cdot K \subseteq N$ and $J \cdot K \subseteq N$. □

Proposition 1

(a) Let $\mathfrak{R}$ be a hyperring and N be a hyperideal of $\mathfrak{R}$ with $N \cap r(\mathfrak{R}) \neq \varnothing$. If J and K are r-hyperideals of $\mathfrak{R}$ such that $N \cdot J = N \cdot K$ or $N \cap J = N \cap K$, then $J = K$.

(b) Let $\mathfrak{R}$ be a hyperring and N, M be hyperideals of $\mathfrak{R}$ with $M \cap r(\mathfrak{R}) \neq \varnothing$. If $N \cdot M$ is r-hyperideal of $\mathfrak{R}$, then $N = N \cdot M$. In addition, N is an r-hyperideal.

Proof

(a) Let J and K be r-hyperideals of $\mathfrak{R}$ and $N \cdot J = N \cdot K$. From Lemma 1, since $N \cdot J = N \cdot K \subseteq K$, then $J \subseteq K$. Therefore, $K \subseteq J$. Then, $J = K$.

(b) Let $N \cdot M$ be r-hyperideal of $\mathfrak{R}$ and $M \cap r(\mathfrak{R}) \neq \varnothing$. Since $N \cdot M \subseteq M \cdot N$, then from Lemma 1, $N \subseteq N \cdot M$. Therefore, obviously, $N \cdot M \subseteq N$. Then, $N = N \cdot M$. □

Theorem 2. *Let $I_1, I_2, \ldots, I_n$ be prime hyperideals of $\mathfrak{R}$, which are not comparable. If $\cap_{i=1}^{n} I_i$ is an r-hyperideal, then I_i is an r-hyperideal, for $i = 1, \ldots, n$.*

Proof. Let $r, x \in \mathfrak{R}$ such that $r \cdot x \in I_i$ and $ann(r) = 0$. Let $y \in \cap_{j \neq i} I_j / I_i$. Then, $r \cdot x \cdot y \in \cap_{i=1}^{n} I_i$. We get that $x \cdot y \in \cap_{i=1}^{n} I_i$, since $\cap_{i=1}^{n} I_i$ is an r-hyperideal and $ann(r) = 0$. Thus, $x \cdot y \in I_i$. We conclude that $x \in I_i$, since $y \notin I_i$ and I_i's are prime. Hence, I_i is an r-hyperideal.

Let $\rho{:}\ \mathfrak{R} \longrightarrow S$ be a homomorphism. We investigate whether the image of an r-hyperideal and the inverse image of an r-hyperideal are r-hyperideal. □

Theorem 3. *Let $\rho{:}\ \mathfrak{R} \longrightarrow S$ be a good epimorphism such that $Ker\rho \subseteq N$. If N is an r-hyperideal of $(\mathfrak{R}, +, \cdot)$, then $\rho(N)$ is an r-hyperideal of the hyperring $(S, \oplus, \odot)$.*

Proof. Obviously, $\rho(N)$ is hyperideal of S. Let $b_1 \odot b_2 \in \rho(N)$ and $ann(b_1) = 0_S$, for $b_1, b_2 \in S$. Since ρ is onto, then there exist $a_1, a_2 \in \mathfrak{R}$ such that $b_1 = \rho(a_1)$ and $b_2 = \rho(a_2)$. Then, $b_1 \odot b_2 = \rho(a_1) \odot \rho(a_2) = \rho(a_1 \cdot a_2) = \rho(x) \in f(N)$, for some $x \in N$.

$0 \in \rho(a_1 \cdot a_2) \ominus \rho(x) = \rho(a_1 \cdot a_2 - x)$. Then, there exists $t \in a_1 \cdot a_2 - x$ such that $\rho(t) = 0$. We have $a_1 \cdot a_2 \in t + x \subseteq Ker\rho + N \subseteq N + N \subseteq N$. Thus, $a_1 \cdot a_2 \in N$.

Let $ann(a_1) \neq 0$. Then, there exists $0 \neq c \in \mathfrak{R}$ such that $a_1 \cdot c = 0_{\mathfrak{R}}$. So, $\rho(a_1 \cdot c) = \rho(a_1) \odot \rho(c) = \rho(0_{\mathfrak{R}}) = 0_S$. Since $\rho(c) \neq 0_S$, then $ann(\rho(a_1)) = ann(b_1) = 0_S$, and this is a contradiction. This implies that $ann(a_1) = 0$. Since N is an r-hyperideal, then $a_2 \in N$. Therefore, $b_2 = \rho(a_2) \in \rho(N)$. □

Theorem 4. *Let $\rho{:}\ \mathfrak{R} \longrightarrow S$ be a good monomorphism. If M is an r -hyperideal of $(S, \oplus, \odot)$, then $\rho^{-1}(M) = M^c$ is an r-hyperideal of $(\mathfrak{R}, +, \cdot)$.*

Proof. $\rho^{-1}(M)$ is a hyperideal [37]. Let $r \cdot x \in \rho^{-1}(M)$ and $ann(r) = 0$. Then, $\rho(r) \odot \rho(x) = \rho(r \cdot x) \in M$. If $ann(\rho(r)) \neq 0$, then there exists $0 \neq \rho(s) \in M$ such that $\rho(r) \odot \rho(s) = \rho(r \cdot s) = 0_S$. This means that there exists a $0 \neq s \in \rho^{-1}(M)$ such that $r \cdot s = 0_{\mathfrak{R}}$. Then, $ann(r) \neq 0$. This is a contradiction, so $ann(\rho(r)) = 0$. Since M is an r-hyperideal, then $\rho(x) \in M$ and therefore $x \in \rho^{-1}(M)$. □

Theorem 5. *Let $\mathfrak{R}$ be a hyperring. The following statements are equivalent:*

(a) $\mathfrak{R}$ is a hyperdomain

(b) The only r-hyperideal of $\mathfrak{R}$ is the zero hyperideal

(c) $ann(x \cdot y) = ann(x) \cup ann(y)$, for each $x, y \in \mathfrak{R}$

Proof

$(a) \Rightarrow (b)$ Let us suppose that $\mathfrak{R}$ is hyperdomain and $(0) \neq N$ is a proper hyperideal of $\mathfrak{R}$. Then, there exists $0 \neq r \in N$. Since $\mathfrak{R}$ is a hyperdomain, then we have $ann(r) = 0$. This contradicts the hyperideal of being proper. Then, the zero hyperideal is the only r-hyperideal.

$(b) \Rightarrow (c)$ From Corollary 1 (f), $ann(x)$ is an r-hyperideal. Assume that the zero hyperideal is the only r-hyperideal of $\mathfrak{R}$. Hence, $ann(x) = 0$. So $ann(x \cdot y) = ann(x) \cup ann(y) = 0$, for every $x, y \in \mathfrak{R}$.

$(c) \Rightarrow (a)$ Let $x \cdot y = 0$, $x, y \in \mathfrak{R}$. Thus, $ann(x \cdot y) = ann(0) = ann(x) \cup ann(y) = \mathfrak{R}$. This means $1_{\mathfrak{R}} \in ann(x) \cup ann(y)$. Then, $x = \{0\}$ or $y = \{0\}$. Hence, $\mathfrak{R}$ is a hyperdomain. □

Proposition 2

(a) *Let $\mathfrak{R}$ be a hyperring and $x, y \in \mathfrak{R}$ with $1_{\mathfrak{R}} \in x + y$. Then, $N = ann(x) + ann(y)$ is an r-hyperideal.*

(b) *Let $\mathfrak{R}$ be a reduced hyperring, $I \in Min(\mathfrak{R})$ and $e \in \mathfrak{R}$ be an idempotent element. We infer that $N = I + ann(e)$ is an r-hyperideal.*

Proof

(a) Let $a \cdot b \in N$ and $ann(a) = 0$. Then, there exists $r \in ann(x)$ and $s \in ann(y)$ such that $a \cdot b \in r + s$. Obviously, $r \cdot x = 0$ and $s \cdot y = 0$. We get that $a \cdot b \cdot x \cdot y \in (r + s) \cdot (x \cdot y) = r \cdot (x \cdot y) + s \cdot (x \cdot y) = y \cdot (x \cdot r) + x \cdot (y \cdot s) = b \cdot 0 + x \cdot 0 = 0$ and by our assumption, $b \cdot x \cdot y = 0$. Thus, $b \cdot x \in ann(y)$ or $b \cdot y \in ann(x)$. Hence, $b = b \cdot 1_{\mathfrak{R}} \in b \cdot (x + y) = (b \cdot x)(b \cdot y) \subseteq ann(x) + ann(y) = N$ and therefore $b \in N$.

(b) Let $r \cdot x \in N$ and $ann(r) = 0$. We have $r \cdot x \in a + b$ such that $a \in I$ and $b \in ann(e)$. Since $I \in Min(\mathfrak{R})$, then I is an r -hyperideal. There is $y \notin I$ such that $a \cdot y = 0$. $e \cdot y \cdot r \cdot x \in e \cdot y \cdot (a + b) = (e \cdot y \cdot a) + (e \cdot y \cdot b) = 0$. Since $\mathfrak{R}$ is a reduced hyperring, then every idempotent element is 0. By our assumption, $e \cdot y \cdot x \in I$. Since $y \notin I$, then $e \cdot x \in I$. We write $x \in x + (e \cdot x) - (e \cdot x)$, $x \in (e \cdot x) + x - (e \cdot x) = e \cdot x + x \cdot (1_{\mathfrak{R}} - e) \subseteq I + ann(e) = N$. Then, N is an r-hyperideal. □

Corollary 2

(a) *If there is a hyperideal K of $\mathfrak{R}$ such that $ann(N) + ann(M) = ann(K)$, for N, M hyperideals, then $ann(N) + ann(M)$ is r-hyperideal.*

(b) *The direct sum of r-hyperideals is r-hyperideal. Therefore, if $N = M \oplus K$ and N is r-hyperideal, then M and K are r-hyperideals*

(c) *Let $\mathfrak{R}$ be a reduced hyperring. Then, $soc(\mathfrak{R})$ is an r-hyperideal. Therefore, there is a hyperideal M of $\mathfrak{R}$ such that $soc(\mathfrak{R}) = M^c$.*

Proposition 3. *Let N be a hyperideal of $\mathfrak{R}$. N is a pr-hyperideal if and only if $\sqrt{N}$ is r-hyperideal.*

Proof. Similarly with [33], let N be pr-hyperideal, $x \cdot y \in \sqrt{N}$ and $ann(x) = 0$. There is $n \in \mathbb{N}$ such that $(x \cdot y)^n \in N$. Hence, $x^n \cdot y^n \in N$ and $ann(x^n) = 0$. Since N is pr-hyperideal, then $y^{nm} \in N$, for $m \in \mathbb{N}\} = \sqrt{0}$. Obviously, $\sqrt{0}$ is r-hyperideal. Then, $nil(\mathfrak{R})$ is also r-hyperideal.

From Theorem 1 and Proposition 3, we obtain the next corollary. □

Corollary 3. *Let N be a hyperideal of $\mathfrak{R}$. The following statements are equivalent:*

(a) *N is pr-hyperideal*

(b) *$(r \cdot \mathfrak{R}) \cap \sqrt{N} = r \cdot \sqrt{N}$, for any $r \in r(\mathfrak{R})$*

(c) *$\sqrt{N} = \sqrt{(N\colon r)}$, for any $r \in r(\mathfrak{R})$ and $r \notin N$*

(d) *$N = M^c$, M is a primary hyperideal of $Q(\mathfrak{R})$*

Theorem 6. *Every z^0-hyperideal is an r-hyperideal.*

Proof. Similar to [33], let $x \cdot y \in N$ and $ann(x) = 0$. Then, $ann(x \cdot y) = ann(y)$. Since N is z^0-hyperideal, then $y \in N$ and N are an r-hyperideal. □

Theorem 7. *Every hyperideal which contains zero divisors is contained by a prime r-hyperideal.*

Proof. It is trivial. □

Theorem 8. *If N is a hyperideal and $I \in Min(N)$ in hyperring $\mathfrak{R}$, then I is an r-hyperideal.*

Proof. Let $x \cdot y \in I$ and $ann(x) = 0$. There exists $a \notin I$ and $n \in \mathbb{N}$ such that $a \cdot (x \cdot y)^n = a \cdot x^n \cdot y^n \in N$. Since N is r-hyperideal and $ann(x^n) = 0$, then $a \cdot y^n \in N \subseteq I$. Since $a \notin I$, then $y^n \in I$. Thus, $y \in I$. □

Proposition 4. *Let N be an r-hyperideal and $N \subseteq M$. If M/N is r-hyperideal of $\mathfrak{R}/N$, then M is r-hyperideal of $\mathfrak{R}$.*

Proof. Let be r-hyperideal of $\mathfrak{R}/N$, $(a + N) \cdot (b + N) \subseteq M/N$ and $ann(a + N) = 0$ for $a + N, b + N \in \mathfrak{R}/N$.

$(a + N) \cdot (b + N) \subseteq a \cdot (b + N) + N \cdot (b + N) \subseteq (a \cdot b) + (a \cdot N) + (N \cdot b) + N \subseteq (a \cdot b) + N \subseteq M + N$. So, $a \cdot b \in M$. Since $ann(a) = 0$ and $b \in M$, then M is an r-hyperideal.

Remember that if M and $\mathfrak{R}$ are the only hyperideals of $\mathfrak{R}$ that contain M, a proper hyperideal M of $\mathfrak{R}$ is said to be a maximal hyperideal of $\mathfrak{R}$. Moreover, in a commutative unitary hyperring, each maximal hyperideal is a prime hyperideal (for more details, see [41]). In the following, we investigate the maximal hyperideal and r-hyperideal relation. □

Proposition 5. *Every maximal r-hyperideal is a prime hyperideal.*

Proof. Obvious from [33]. □

Proposition 6. *If every prime hyperideal of $\mathfrak{R}$ is an r-hyperideal, every maximal hyperideal of $\mathfrak{R}$ is an r-hyperideal.*

Proof. Suppose that M is a maximal hyperideal of $\mathfrak{R}$. We know that every maximal hyperideal is a prime hyperideal. Let $a \cdot b \in M$ with $ann(a) = 0$. Then, $b \in (M\colon a)$. Since M is maximal, then $M = (M\colon a)$ and $b \in M$. □

Proposition 7. *Let $\mathfrak{R}$ be a reduced hyperring satisfying the Property A. Then, every maximal r-hyperideal is an z^0-hyperideal.*

Proof. Let I be a maximal r-hyperideal, $I \subseteq z\ d(\mathfrak{R})$. There exists a z^0-hyperideal J such that $I \subseteq J$. Then, J is an r-hyperideal. Since I is maximal, then $I = J$. Thus, I is an z^0-hyperideal. □

4. Generalizations of r-Hyperideals

In this section, we introduce a generalization of r-hyperideals to commutative Krasner hyperrings. $I\ d(\mathfrak{R})$ denotes the hyperideals of $\mathfrak{R}$. Initially, we give the definition of $\phi-$ prime and $\phi-$ primary hyperideals. Similarly, we give the definitions of $\phi - r$-hyperideal, $\phi - pr$-hyperideal, $\phi-$ pure hyperideal, and $\phi-$ von Neumann regular hyperideal in $\mathfrak{R}$. Afterwards, we investigate some properties of $\phi - r$-hyperideal of $\mathfrak{R}$.

Definition 10. Let $\mathfrak{R}$ be a Krasner hyperring, $\phi\colon I\ d(\mathfrak{R}) \longrightarrow I\ d(\mathfrak{R}) \cup \{\varnothing\}$ be a function and $\varnothing \neq N \in I\ d(\mathfrak{R})$. Then, N is said to be a $\phi-$ prime (resp. $\phi-$ primary) hyperideal of $\mathfrak{R}$ if whenever $r, s \in \mathfrak{R}$ and $r \cdot s \in N - \phi(N)$, then $r \in N$ or $s \in N$ (resp. $r \in N$ or $s^n \in N$).

Definition 11. Let $\mathfrak{R}$ be a Krasner hyperring, N be a proper hyperideal of $\mathfrak{R}$ and $\phi\colon I\ d(\mathfrak{R}) \longrightarrow I\ d(\mathfrak{R}) \cup \{\varnothing\}$ be a function. N is called $\phi - r$-hyperideal (resp., $\phi - pr$-hyperideal) if $a \cdot b \in N - \phi(N)$ with $ann(a) = 0$ implies that $b \in N$ (resp., $b^n \in N$), for $a, b \in \mathfrak{R}$.

Example 6. Let J be a $\phi - r$-hyperideal of $\mathfrak{R}$. Then,

(i) If $\phi(J) = \varnothing$, for all $J \in I\ d(\mathfrak{R})$, then J is called $\phi_\varnothing - r$-hyperideal of $\mathfrak{R}$ (r-hyperideal).
(ii) If $\phi(J) = 0$, for all $J \in I\ d(\mathfrak{R})$, then J is called $\phi_0 - r$-hyperideal of $\mathfrak{R}$ (weakly r-hyperideal).
(iii) If $\phi(J) = J$, for all $J \in I\ d(\mathfrak{R})$, then J is called $\phi_1 - r$-hyperideal of $\mathfrak{R}$ (any hyperideal).
(iv) If $\phi(J) = J^n$, for all $J \in I\ d(\mathfrak{R})$ and $n \geq 2$, then N is called $\phi_n - r$-hyperideal (n-almost r-hyperideal). If $n = 2$, then J is called almost r-hyperideal of $\mathfrak{R}$.
(v) If $\phi(J) = \cap_{i=1}^{\infty} J^i$, for all $J \in I\ d(\mathfrak{R})$, then J is called $\phi_\omega - r$-hyperideal ($\omega - r$-hyperideal) of $\mathfrak{R}$.

Definition 12. Let $\mathfrak{R}$ be a Krasner hyperring, N be a proper hyperideal of $\mathfrak{R}$ and $\phi\colon I\ d(\mathfrak{R}) \longrightarrow I\ d(\mathfrak{R}) \cup \{0\}$ be a function.

(1) N is called $\phi-$ pure hyperideal if there exists $b \in N$ such that $a = a \cdot b$, for every $a \in N - \phi(N)$
(2) N is called $\phi-$ von Neumann regular hyperideal if there exists $b \in N$ such that $a = a^2 \cdot b$, for every $a \in N - \phi(N)$

Theorem 9. *Let $N \in I\ d(\mathfrak{R})$. Then, the following statements hold:*

(i) Let ψ_1 and ψ_2 be two functions. $\psi_{1,2}\colon I\ d(\mathfrak{R}) \longrightarrow I\ d(\mathfrak{R}) \cup \{\varnothing\}$ such that $\psi_1 \leq \psi_2$. If N is a $\psi_1 - r$-hyperideal, then N is a $\psi_2 - r$-hyperideal
(ii) Every $\phi - r$-hyperideal is a $\phi - pr$-hyperideal
(iii) N is an r-hyperideal $\Rightarrow$N is a weakly r-hyperideal $\Rightarrow$ N is an $\omega - r$-hyperideal $\Rightarrow$ N is an $(n+1)$-almost r-hyperideal $\Rightarrow$N is an n-almost r-hyperideal $\Rightarrow$N is an almost r-hyperideal
(iv) Every $\phi-$ pure hyperideal is a $\phi - r$-hyperideal
(v) Every $\phi-$ von Neumann regular hyperideal is a $\phi - r$-hyperideal

Proof. Straightforward. □

Theorem 10. *Let $\mathfrak{R}$ be a Krasner hyperring, N be a proper hyperideal of $\mathfrak{R}$. Then, the following statements are equivalent:*

(i) N is $\phi - r$-hyperideal
(ii) $(N\colon r) = N \cup (\phi(N)\colon r)$, $r \in r(\mathfrak{R})\backslash N$
(iii) $(N\colon r) = N$ or $(N\colon r) = (\phi(N)\colon r)$, $r \in r(\mathfrak{R})/N$

Proof $(i) \Rightarrow (ii)$ We know that $N \subseteq (N\colon r)$ and $(\phi(N)\colon r) \subseteq (N\colon r)$. Let $s \in (N\colon r)$. Then, $r \cdot s \in N$. If $r \cdot s \in \phi(N)$, then $s \in (\phi(N)\colon r)$. On the other hand, if $r \cdot s \notin r \cdot s \notin fa$, since N is $\phi - r$-hyperideal, then $s \in N$.

$(ii) \Rightarrow (iii)$ It is evident.

$(iii) \Rightarrow (i)$ Let $r, s \in \mathfrak{R}$ such that $r \cdot s \in N - \phi(N)$ and r be regular. Then, $(N\colon r) = N$ or $(N\colon r) = (\phi(N)\colon r)$. □

Case 1. .Assume that $(N\colon r) = N$. Since $r \cdot s \in N$, then $s \in (N\colon r) = N$.

Case 2. .Assume that $(N\colon r) = (\phi(N)\colon r)$. Since $r \cdot s \in N$, then $s \in (N\colon r)$. Thus, $r \cdot s \in \phi(N)$. This is a contradiction.

Definition 13. Let N be a proper hyperideal of $\mathfrak{R}$, J and K be hyperideals of $\mathfrak{R}$ such that $J \cdot K \subseteq N$, $J \cdot K \nsubseteq \phi(N)$ and $ann(J) = 0$ implies that $K \subseteq N$. Then, N is called a strongly $\phi - r$-hyperideal.

Theorem 11. *Every strongly $\phi - r$-hyperideal is an $\phi - r$-hyperideal.*

Proof .Similar to [36], let $a, b \in \mathfrak{R}$ such that $a \cdot b \in N - \phi(N)$ and $ann(a) = 0$. Then, $\langle a \rangle \cdot \langle b \rangle \subseteq N - \phi(N)$. Obviously, $ann(\langle a \rangle) = 0$. Since N is strongly $\phi - r$-hyperideal, then $< b > \subseteq N$. Therefore, $b \in N$. □

Theorem 12. *Let $\mathfrak{R}$ satisfy the s.a.c., N be a proper hyperideal of $\mathfrak{R}$ and $\phi(N)$ be an r-hyperideal of $\mathfrak{R}$. N is $\phi - r$-hyperideal if and only if for every hyperideal J and K of $\mathfrak{R}$ such that J is finitely generated, whenever $J \cdot K \subseteq N$, $J \cdot K \nsubseteq \phi(N)$, and $ann(J) = 0$, implies that $K \subseteq N$.*

Proof. ⇒: Let N be a $\phi - r$-hyperideal, $\phi(N)$ be an r-hyperideal, J be finitely generated, J and K be hyperideals of $\mathfrak{R}$ such that $J \cdot K \subseteq N$ and $J \cdot K \nsubseteq \phi(N)$. Assume that $K \nsubseteq N$. Then, there exists an element $0 \neq b \in K - N$. Since $\mathfrak{R}$ satisfy the s.a.c and J is finitely generated, then there is an element $a \in J$ with $ann(J) = ann(a) = 0$. Let $a \cdot b \in N$. If $a \cdot b \notin \phi(N)$, since N is $\phi - r$-hyperideal, then $b \in N$ which is a contradiction. If $a \cdot b \in \phi(N)$, since $\phi(N)$ is an r-hyperideal, then $b \in \phi(N)$ which is a contradiction. Thus, $K \subseteq N$.

⇐: It is obvious.

Now, we investigate the relation between the hyperideal N and the set of all zero divisors, while $\phi(N)$ is an r-hyperideal of $\mathfrak{R}$. □

Theorem 13. *Let $\phi(N)$ be an r-hyperideal of $\mathfrak{R}$ and N be a proper hyperideal of $\mathfrak{R}$. If N is $\phi - r$-hyperideal, then $N \subseteq z\ d(\mathfrak{R})$.*

Proof. Let $N \nsubseteq z\ d(\mathfrak{R})$. Then, there exists an element $a \in N - z\ d(\mathfrak{R})$. So, $ann(a) = 0$. We have $a = a \cdot 1_{\mathfrak{R}} \in N$. If $a \notin \phi(N)$, then $1_{\mathfrak{R}} \in N$, which is a contradiction. If $a \in \phi(N)$, then $1_{\mathfrak{R}} \in \phi(N) \subseteq N$, since $\phi(N)$ is an r-hyperideal. □

Theorem 14. *Let $\phi(N)$ be an r-hyperideal of $\mathfrak{R}$ and N be a prime hyperideal of $\mathfrak{R}$. N is $\phi - r$-hyperideal if and only if $N \subseteq z\ d(\mathfrak{R})$.*

Proof. Let $\phi(N)$ be an r-hyperideal of $\mathfrak{R}$ and N be a prime hyperideal.

⇒: It is obvious from the previous theorem.

⇐: Let $a \cdot b \in N - \phi(N)$ and $ann(a) = 0$. Since N is a prime hyperideal, $a \in N$ or $b \in N$. Since $N \subseteq z\ d(\mathfrak{R})$, then $a \notin N$. Therefore, $b \in N$. □

In a hyperideal N, nilpotent elements are also zero divisors. So we can change the theorem as if N is a hyperideal of which elements are nilpotent, then N is $\phi - r$-hyperideal if and only if N is ϕ-primary hyperideal. □

Theorem 15. *If N is $\phi - r$-hyperideal, then $N - \phi(N) \subseteq z\ d(\mathfrak{R})$.*

Proof. Let us suppose that $N - \phi(N) \nsubseteq z\ d(\mathfrak{R})$. Then, there exists an element a in $N - \phi(N)$ such that $ann(a) = 0$. Since $a = a \cdot 1 \in N - \phi(N)$ and N is $\phi - r$-hyperideal, then $1 \in N$. That is a contradiction. □

Theorem 16. *Let $\phi(N)$ be an r-hyperideal. N is $\phi - r$-hyperideal if and only if N is an r-hyperideal.*

Proof. (⇒:) Let $a \cdot b \in N$ and $ann(a) = 0$. If $a \cdot b \in \phi(N)$, since $\phi(N)$ is an r-hyperideal, then $b \in \phi(N) \subseteq N$. If $a \cdot b \notin \phi(N)$, since N is ϕ– r-hyperideal, then $b \in N$.

(⇐:) It is obvious.

Similar to [36], if N is a hyperideal of $\mathfrak{R}$, we define $\phi_N: I\ d(\mathfrak{R}/N) \longrightarrow I\ d(\mathfrak{R}/N) \cup \{\varnothing\}$ by $\phi_N(M/N) = (\phi(M) + N)/N$ for each hyperideal $N \subseteq M$. If $\phi(N) = \varnothing$, $\phi_N(M/N) = \varnothing$. We see that $\phi_N(M/N) \subseteq (M/N)$. □

Theorem 17. *Suppose that $N \subseteq r(\mathfrak{R})$ and M are hyperideals in $\mathfrak{R}$ such that $N \subseteq M$. If M is $\phi - r$-hyperideal of $\mathfrak{R}$, then (M/N) is $\phi_N - r$-hyperideal of $\mathfrak{R}/N$.*

Proof. Suppose that- $(r + N) \cdot (s + N) \subseteq (M/N) - \phi_M(M/N) = (M/N) - (\phi(M) + N)/N$ and $ann_{\mathfrak{R}/N}(r + N) = 0$. Then, $r \cdot s \in M - \phi(M)$ and as $N \subseteq r(\mathfrak{R})$, $ann_{\mathfrak{R}}(r) = 0$. Since M is $\phi - r$-hyperideal, then $s \in M$. Therefore, $s + N \subseteq (M/N)$. □

Theorem 18. *Suppose that N is an r-hyperideal and M is a hyperideal in $\mathfrak{R}$ such that $N \subseteq M$. If (M/N) is r-hyperideal of $\mathfrak{R}/N$, then M is $\phi - r$-hyperideal of $\mathfrak{R}$.*

Proof. Suppose that $r \cdot s \in M - \phi(M)$ with $ann(r) = 0$. If $r \cdot s \in N$, since N is an r-hyperideal, then $s \in N \subseteq M$. If $r \cdot s \in M - N$, then $r \cdot s + N = (r + N) \cdot (s + N) \subseteq (M/N)$. In addition, since $ann(r) = 0$, then $ann(r + N) = 0$. Therefore, since (M/N) is r-hyperideal of $\mathfrak{R}/N$, then $s + N \subseteq (M/N)$. Hence, $s \in M$, as desired.

Let $\phi: I\ d(\mathfrak{R}) \longrightarrow I\ d(\mathfrak{R}) \cup \{0\}$ be a function. If N and M are hyperideals of $\mathfrak{R}$ such that $N \subseteq M$ implies that $\phi(N) \subseteq \phi(M)$, then ϕ is said to preserve the order. □

Theorem 19. *Let A be a subset of $\mathfrak{R}$, $A \neq \varnothing$ and ϕ preserves the order. If N is $\phi - r$-hyperideal such that $A \nsubseteq N$, then $(N: A)$ is $\phi - r$-hyperideal.*

Proof. Suppose that $a \cdot b \in (N: A) - \phi(N: A)$ and $ann(a) = 0$. Since ϕ preserves order, then $\phi(N) \subseteq \phi(N: A)$. We have $a \cdot b \cdot A \subseteq N - \phi(N)$. Therefore, $b \cdot A \subseteq N$ and $b \in (N: A)$. □

Corollary 4. *Let $\{N_i\}_{i \in \Lambda}$ directed collection of ascending chain $\phi - r$-hyperideals of $\mathfrak{R}$ and ϕ preserves the order. Then, $N = \bigcup_{N \in \Lambda} N_i$ is $\phi - r$-hyperideal.*

Proof. Let $a \cdot b \in N - \phi(N)$ and $ann(a) = 0$. Then, $a \cdot b \in N_k$ and $a \cdot b \notin \phi(N_N)$, for $k \in \Lambda$. Since N_k is $\phi - r$-hyperideal, then $b \in N_k \subseteq N$. □

5. Conclusion

In this paper, some properties of r-hyperideals are investigated in the commutative Krasner hyperrings. For r-hyperideals and pr-hyperideals, many characterizations are introduced, and some examples are given. The theoretical point of view of $\phi - r$-hyperideals which is the generalization of $\phi - r$-ideal is examined. After this study, one can introduce and examine them in Krasner (m, n)-hyperrings and do some studies on their properties. This approach can also be used to algebraic structures like fuzzy sets and Zariski topology. We suggest open problems with δ-r-hyperideals, 2-absorbing r-hyperideals, and r-subhypermodules.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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