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Existence and Physical Properties of Gradient Ricci–Yamabe Solitons

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Abstract—We first prove the existence of the gradient Ricci–Yamabe soliton (briefly GRYS) by constructing an explicit example endowed with the Robertson–Walker metric. Then we focus on the physical properties of the gradient Ricci–Yamabe solitons satisying Einstein's field equations, under the assumptions of different subspaces of Gray's decompositions. For instance, we prove that if a GRYS space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field, belongs to the subspaces $\mathcal{B}$ and $\mathcal{B}'$, then it is a Robertson–Walker space-time with vanishing shear and vorticity. Moreover, its possible local cosmological structures are of Petrov types I, D, or O. Finally, we obtain the equations of state of a perfect-fluid space-time admitting the GRYS whose velocity field is a unit-timelike Killing vector field.

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1. INTRODUCTION

In the last half of the 1900s, some flow equations defined by various systems of partial differential equations began to be defined and studied frequently. Some of these are the Riemann flow, the Ricci flow and the Yamabe flow introduced by Hamilton [21], the Ricci-Bourguignon flow introduced by Bourguignon [6] and the Ricci–Yamabe flow introduced by Güler and Crasmareanu [19]. In the last 30 years, many mathematicians have become interested in self-similar solutions of these geometric flows. Some groundbreaking examples in the literature include the following: Perelman [33] defined Ricci solitons, which are self-similar solutions of the Ricci flow, and showed that compact Ricci solitons have a gradient structure. Blaga [3] extended the study of Ricci solitons to almost η-Ricci solitons in $(LCS)_n$ manifolds. Also, in [4], Blaga studied the Ricci solitons in general relativistic aspects. Güler and Ünal [20] proved the existence of nontrivial gradient Yamabe solitons on generalized Robertson–Walker space-times, standard static space-times, Walker manifolds and pp-wave space-times. Blaga and Tastan [5] investigated almost Ricci-Bourguignon solitons admitting some special potential vector fields. Moreover, in [25], by using the Hodge-de Rham decomposition of the potential function, it is shown that under some conditions a compact gradient almost Ricci–Yamabe soliton is isometric to the Euclidean sphere $S^n(r)$.

The main subject of this article are self-similar solutions of the Ricci–Yamabe flow defined in [19], which are called gradient Ricci–Yamabe solitons in [12]: Let (M^n, g) be a (pseudo)-Riemannian manifold, and $\psi : M \to \mathbb{R}$ be a differentiable function. Then the sextuple $(M, g, \psi, \lambda, \alpha, \beta)$ is said to be a gradient Ricci–Yamabe soliton (briefly GRYS) if the following equation

$$\mathrm{Hess}\psi + \alpha \mathrm{Ric} = \left(\lambda - \frac{1}{2}\beta\tau\right) \mathrm{g}, \qquad (1)$$

holds, where Ric and τ are the Ricci curvature and the scalar curvature of (M, g), respectively, Hessψ is the Hessian of ψ, and $\lambda, \alpha, \beta \in \mathbb{R}$ [12]. The manifold satisfying Eq. (1) is simply referred to GRYS of (α, β)-type. Particularly, the α-Ricci soliton and the β-Yamabe soliton are GRYS of type $(\alpha, 0)$, $(0, \beta)$-type, respectively [12]. A GRYS is called expanding, steady or shrinking, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. For recent studies, we refer to [9–11, 13, 24].

This paper is organized as follows: In Section 2, we first construct an explicit example of a GRYS by using the Robertson–Walker space-time metric with spatial coordinates. In Section 3, we investigate the GRYS with the physical point of view. In Theorem 3 we prove that a GRYS-space-time $(M, g, \psi, \lambda, \alpha, \beta)$, in which the gradient of the potential function ψ is a

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unit-timelike torse-forming vector field, is a perfect-fluid space-time whose velocity field is $\nabla\psi$, and the associated scalars given by (29) are constants. Then, in Theorem 3, it is proved that the isotropic pressure and the energy density of this space-time are constants, and the expansion scalar and the acceleration vector of the fluid vanish. In Section 4, we examine the cases where GRYS-space-time belongs to subspaces in Gray's decomposition, respectively. For instance, in Theorem 6, we prove that if a GRYS-space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field, belongs to the subspaces $\mathcal{B}$ and $\mathcal{B}'$, then it is a Robetson—Walker space-time with vanishing shear and vorticity. Moreover, its possible local cosmological structures are of Petrov type I, D, or O. Finally, in Theorem 9, it is shown that If a perfect fluid space-time satisfying Einstein's field equations admits a $(M, g, \psi, \lambda, \alpha, \beta)$ GRYS whose velocity field is a unit-timelike Killing vector field $\nabla\psi$, then it represents a phantom era.

2. EXISTENCE OF GRYS: AN EXPLICIT EXAMPLE

Let (r, θ, φ) be spatial coordinates, and $A(t)$ denote the expansion scalar of cosmic time. Then the flat Robertson—Walker space-time metric is

$$ds^2 = dt^2 - A^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \quad (2)$$

Assume that $(x^0, x^1, x^2, x^2) = (t, r, \theta, \varphi)$ represent the standard coordinates of $\mathbb{R}^4$. Then the only nonvanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are obtained as follows:

$$\Gamma^0_{11} = AA', \quad \Gamma^0_{22} = AA'r^2,$$
$$\Gamma^0_{33} = AA'r^2\sin^2\theta,$$
$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{A'}{A},$$
$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta,$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r},$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta,$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \quad (3)$$
$$\mathrm{Ric}_{00} = 3\frac{A''}{A},$$
$$\mathrm{Ric}_{11} = -[A'' + 2(A')^2],$$
$$\mathrm{Ric}_{22} = -r^2[A'' + 2(A')^2],$$
$$\mathrm{Ric}_{33} = -r^2\sin^2\theta[A'' + 2(A')^2], \quad (4)$$

and the components which can be obtained from these by symmetry properties. Also, the scalar curvature is given by

$$\tau = 6\Big[\frac{A''}{A} + \Big(\frac{A'}{A}\Big)^2\Big], \quad (5)$$

where the prime denotes derivatives with respect to t. To construct some explicit examples of a GRYS with the potential function ψ, now we evaluate the components of the Hessian tensor by using the metric (9) as follows:

$$(\mathrm{Hess}\psi)_{00} = \psi'',$$
$$(\mathrm{Hess}\psi)_{01} = \psi'_r - \frac{A'}{A}\psi_r,$$
$$(\mathrm{Hess}\psi)_{02} = \psi'_\theta - \frac{A'}{A}\psi_\theta,$$
$$(\mathrm{Hess}\psi)_{03} = \psi'_\varphi - \frac{A'}{A}\psi_\varphi,$$
$$(\mathrm{Hess}\psi)_{11} = \psi_{rr} - AA'\psi',$$
$$(\mathrm{Hess}\psi)_{12} = \psi_{r\theta} - \frac{1}{r}\psi_\theta,$$
$$(\mathrm{Hess}\psi)_{13} = \psi_{r\varphi} - \frac{1}{r}\psi_\varphi,$$
$$(\mathrm{Hess}\psi)_{22} = \psi_{\theta\theta} - AA'r^2\psi' + r\psi_r,$$
$$(\mathrm{Hess}\psi)_{23} = f_{\theta\varphi} - \mathrm{ctg}\,\theta\psi_\theta,$$
$$(\mathrm{Hess}\psi)_{33} = \psi_{\varphi\varphi} - r^2\sin^2\theta AA'\psi'$$
$$+ r\sin^2\theta\psi_r + \sin\theta\cos\theta\psi_\theta, \quad (6)$$

where subscripts denote partial derivatives with respect to the corresponding variables. Then, the fundamental equation of GRYS given by

$$\mathrm{Hess}\psi + \alpha\mathrm{Ric} = \left(\lambda - \frac{1}{2}\beta\tau\right)\mathrm{g},$$

yields the following system of partial differential equations to solve the potential function ψ:

$$\psi'' = \lambda - 3\frac{\beta}{t^2},$$
$$\psi' = -\frac{2}{t} + t\Big(\lambda - 3\frac{\beta}{t^2}\Big), \quad (7)$$

by choosing the soliton constant $\alpha = 1$, the expansion scalar $A(t) = t$, and the potential function ψ as a function of only t.

Therefore, the solution of the system (7) yields

$$\psi = \frac{\lambda}{2}t^2 - \ln|t| + \mu, \quad \text{where} \quad \mu \in \mathbb{R},$$
$$\beta = -\frac{1}{3}, \quad \text{and} \quad \lambda \in \mathbb{R}. \quad (8)$$

Hence we can state that:

Theorem 1. *Let (M^4, g) be a manifold endowed with the Robertson–Walker metric*

$$ds^2 = g_{ij}dx^i dx^j = dt^2 - t^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \quad (9)$$

where t denotes time, $\{r, \theta, \varphi\}$ are spatial coordinates, and $(x^0, x^1, x^2, x^2) = (t, r, \theta, \varphi)$. Then $(M^4, g, \psi = \frac{\lambda}{2}t^2 - \ln|t| + \mu, \lambda, \alpha = 1, \beta = -\frac{1}{3})$ is a nontrivial GRYS-space-time, where λ and μ are arbitrary soliton constants.

In the particular case if we choose the expansion scalar $A(t) = c = \text{const}$, the metric (9) becomes Ricci-flat. This metric is simply a Minkowski metric and thus a trivial example of a Ricci-flat Robertson–Walker space-time. But still the fundamental equation of GRYS space-time can be solved with a potential function being a linear function of t as follows:

Corollary 1. *Let (M^4, g) be a manifold endowed with the Minkowski metric*

$$ds^2 = g_{ij}dx^i dx^j = dt^2 - c^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (10)$$

where $c \in \mathbb{R}$, t denotes time, $\{r, \theta, \varphi\}$ are spatial coordinates, and $(x^0, x^1, x^2, x^2) = (t, r, \theta, \varphi)$. Then $(M^4, g, \psi = \mu t + \gamma, \lambda = 0, \alpha, \beta)$ is a Ricci-flat and steady GRYS space-time, where μ, α, β and λ are arbitrary soliton constants.

3. GRYS FROM A PHYSICAL POINT OF VIEW

In general theory of relativity, perfect fluids are of great importance for being special solutions of Einstein's field equations having vanishing shear stresses, viscosity and heat conduction compatible with the Bianchi identities. It is known that perfect fluids are the best model for the matter content of the isotropic universe. For these reasons, the geometric and physical properties of perfect fluids are very important areas of study.

An n-dimensional Lorentzian manifold (M^n, g) satisfying the Ricci condition

$$\text{Ric} = ag + b\eta \otimes \eta, \quad (11)$$

where a, b are some scalar fields, U is a vector field such that $\eta(X) = g(X, U)$ for all $X \in \chi(M)$, and $g(U, U) = -1$ is said to be a perfect-fluid space-time. The timelike vector field U is defined as the velocity vector field of the perfect fluid space-time.

Equivalently, Eq. (11) can be written as

$$QX = aX + b\eta(X)U, \quad (12)$$

where Q is a $(1,1)$-type Ricci operator defined by $\text{Ric}(X, Y) = g(QX, Y)$, for any vector fields X, Y. Equation (12) also yields the scalar curvature as

$$\tau = an - b. \quad (13)$$

Many studies have been done on perfect fluids in the recent years by De and Suh [8], Mallick and De [26], Güler and Altay Demirbağ [17, 18] with many different points of view. The equation of state of a perfect fluids in an isotropic universe corresponds to Robertson–Walker space-times. They satisfy Einstein's field equations and describe isotropic, expanding or contracting, homogeneous universes. Since the scale factor of the universe is derived as a function of time, the metric of the Robertson Walker space-time can be expressed as a certain warped product metric (see Section 2). In [1], Alias introduced the notion of a generalized Robertson Walker space-time (M^4, g) as a warped product endowed with the metric

$$g = -\mathrm{d}t^2 \oplus f(t)^2 g^*, \quad (14)$$

where g^* is the metric of the $(n-1)$-dimensional Riemannian manifold M^*, and $f : I \to (0, \infty)$ is a smooth function. If, in particular, g^* is the metric of a 3-dimensional Riemannian manifold of constant curvature, then (M^4, g) is called a Robertson Walker space-time. Because of this reason, a generalized Robertson–Walker space-time is a generalization of the Robertson–Walker space-time. Moreover, every Robertson–Walker space-time is a perfect-fluid one, but the converse statement is not generally true. In dimension 4, a generalized Robertson–Walker space-time is a perfect-fluid one if and only if it is a Robertson–Walker space-time. For more, we refer to [7, 27, 29].

The energy momentum tensor of perfect-fluid space-times is described as

$$T(X, Y) = (\sigma + p)B(X)B(Y) + pg(X, Y), \quad (15)$$

where $g(X, U) = B(X)$, $B(U) = -1$, for all vector fields X, Y, with σ the energy density, and p being the isotropic pressure. The unit timelike vector field U denotes the velocity field [32]. While the energy momentum tensor T gives information on the physical properties of space-time, the Ricci tensor controls the geometry of space-time. In general relativity, they are associated to each other by Einstein's field equations [32] given by

$$\text{Ric}(X, Y) - \frac{\tau}{2}g(X, Y) = \kappa T(X, Y) \quad (16)$$

where Ric is the $(0,2)$-type Ricci tensor, τ is the scalar curvature, and κ is the gravitational constant. By virtue of Eq. (16), the energy momentum tensor is a divergence-free and symmetric tensor [32]. By using (15) into (16) and comparing with (11), we obtain the relations:

$$a = \frac{\kappa(p-\sigma)}{2-n}, \quad b = \kappa(p+\sigma). \tag{17}$$

Moreover, σ and p are bounded by the equation $p = p(\sigma, T_0)$, where the absolute temperature is indicated by T_0 in the perfect fluid. If $p = p(\sigma)$, the perfect fluid is said to be isentropic, [23], and if $p = \sigma$, the perfect fluid is said to be stiff matter [38].

If we take a covariant derivative of (12), we obtain

$$(\nabla_Y \mathrm{Q})(X) = da(Y)X + db(Y)\eta(X)U + b[(\nabla_Y \eta)(X)U + \eta(X)(\nabla_Y U)], \tag{18}$$

for all $X, Y \in \chi(M)$.

Now, assume that the velocity vector field U of a perfect-fluid space-time is a unit torse-forming vector field, introduced by Yano [39]. Then we have

$$\nabla_X U = \Phi[X + \eta(X)U], \tag{19}$$

where $\Phi \in \mathbb{R}$. Equivalently, for the dual form η, we have

$$(\nabla_X \eta)(Y) = \Phi[g(X,Y) + \eta(X)\eta(Y)] = (\nabla_Y \eta)(X) \tag{20}$$

Thus $d\eta = 0$, i.e., η is a closed 1-form.

By putting $Y = U$ in (11), we have

$$\mathrm{Ric}(X,U) = (a-b)\eta(X), \quad \forall X \in \chi(M). \tag{21}$$

It is known that the space-time (M,g) is a generalized Robertson—Walker space-time if and only if it admits a unit timelike torse-forming vector field given as in (19) and (21) [27],

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{22}$$

By putting $Z = U$ into (22) and using (19), we obtain

$$R(X,Y)U = \Phi^2[\eta(Y)X - \eta(X)Y]. \tag{23}$$

Contracting (23) over X, we get

$$\mathrm{Ric}(Y,U) = (n-1)\Phi^2 \eta(Y). \tag{24}$$

Thus, comparing (21) and (24), we have

$$\Phi^2(n-1) = a - b, \tag{25}$$

which yields that $a - b$ is a constant.

Now, assume that the gradient of the potential function ψ of the GRYS is a unit-timelike torse-forming vector field, whose dual 1-form is η. Then by (19) we have

$$\nabla_X \nabla\psi = \Phi[X + \eta(X)\nabla\psi], \tag{26}$$

which gives us

$$(\mathrm{Hess}\psi)(X,Y) = \Phi[g(X,Y) + \eta(X)\eta(Y)]. \tag{27}$$

Then by using (27) in the fundamental equation (1) of the GRYS, we conclude that

$$\mathrm{Ric} = ag + b\eta \otimes \eta, \tag{28}$$

where

$$a = \frac{1}{\alpha}\Big[\lambda - \frac{1}{2}\beta\tau - \Phi\Big], \quad \text{and} \quad b = -\frac{\Phi}{\alpha}. \tag{29}$$

By virtue of (28) and (29), we conclude that:

Theorem 2. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS-space-time. If the gradient of the potential function ψ is a unit-timelike torse-forming vector field, then (M,g) becomes a perfect-fluid space-time whose velocity field is $\nabla\psi$, and the associated scalars given by* (29) *are constants.*

In view of Theorem 2, taking into account that $U = \nabla\psi$, Eq. (18) yields

$$(\nabla_Y \mathrm{Q})(X) = b[(\nabla_Y \eta)(X)\nabla\psi + \eta(X)(\nabla_Y \nabla\psi)], \tag{30}$$

for all $X, Y \in \chi(M)$. Putting $X = \nabla\psi$ into (30), we get

$$(\nabla_Y \mathrm{Q})(\nabla\psi) = -b\nabla_Y \nabla\psi. \tag{31}$$

Moreover, putting $Y = \nabla\psi$ into (27), we get

$$\nabla_{\nabla\psi}\nabla\psi = 0; \tag{32}$$

i.e., the integral curves of the velocity field $\nabla\psi$ are geodesics.

Since $a, b \in \mathbb{R}$, by (17), for a 4-dimensional perfect-fluid space-time (M,g), the isotropic pressure and the energy density are constants and are given by

$$\sigma = \frac{2a+b}{2\kappa}, \quad \text{and} \quad p = \frac{b-2a}{2\kappa}. \tag{33}$$

It is very-well known [32] that the energy and force equations for a perfect-fluid space-time with the velocity field $\nabla\psi$ are given by

$$(\nabla\psi)\sigma = g(\nabla\sigma, \nabla\psi) = -(\sigma+p)\mathrm{div}(\nabla\psi), \tag{34}$$

and

$$(\sigma+p)(\nabla_{\nabla\psi}\nabla\psi) = -\nabla_\perp p = -\nabla p - g(\nabla p, \nabla\psi)\nabla\psi, \tag{35}$$

where the spatial pressure gradient $\nabla_\perp p$ is the component of ∇p orthogonal to $\nabla\psi$.

Since σ is constant, it follows from (34) that either $\sigma + p = 0$ or $\mathrm{div}(\nabla\psi) = 0$. Also, since p is constant from (35), we get either $\sigma + p = 0$ or $\nabla_{\nabla\psi}\nabla\psi = 0$. If $\sigma + p = 0$, the matter content satisfies the vacuumlike equation of state, so we may take $\sigma + p \neq 0$. Thus $\mathrm{div}(\nabla\psi) = 0$, i.e., the expansion scalar vanishes. Also, we have $\nabla_{\nabla\psi}\nabla\psi = 0$, i.e., the acceleration vector vanishes. Thus we can state the following:

Theorem 3. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time. If the gradient of the potential function ψ is a unit-timelike torse-forming vector field, then*

1. *the integral curves of the velocity field $\nabla\psi$ are geodesics;*
2. *the isotropic pressure and the energy density of the fluid are constants;*
3. *the expansion scalar and the acceleration vector of the fluid vanish.*

4. GRYS FROM GRAY'S DECOMPOSITIONS POINT OF VIEW

Gray [15] asserted that the covariant derivative of the Ricci tensor $\nabla\mathrm{Ric}$ can be decomposed into $\mathcal{O}(n)$-invariant terms. This decomposition introduced the six classes of Einstein-like manifolds whose Ricci tensors satisfy some condition on each subspace. In [30, 34], Mantica et al. obtained the form of the Ricci tensor in all $\mathcal{O}(n)$-invariant subspaces in the generalized Robertson Walker space-times. According to this study, in all cases except one, the space-time reduces to an Einstein space or has the matter content of a perfect fluid.

In [31], it is shown that $\nabla\mathrm{Ric}$ can be written as

$$(\nabla_Z\mathrm{Ric})(V,W) = \tilde{R}(Z,V)W + \alpha(Z)g(V,W) + \beta(V)g(Z,W) + \beta(W)g(V,Z), \tag{36}$$

for all vector fields Z, V, W, where

$$\alpha(Z) = \frac{n}{(n-1)(n+2)}\nabla_Z\tau,$$
$$\beta(Z) = \frac{n-2}{2(n-1)(n+2)}\nabla_Z\tau, \tag{37}$$

with $\tilde{R}(Z,V)W = \tilde{R}(Z,W)V$ being the traceless tensor that can be written as a sum of its orthogonal components,

$$\tilde{R}(Z,V)W = \frac{1}{3}[\tilde{R}(Z,V)W - \tilde{R}(V,Z)W] + \frac{1}{3}[\tilde{R}(Z,V)W - \tilde{R}(W,Z)V] + \frac{1}{3}[\tilde{R}(Z,V)W + \tilde{R}(V,W)Z + \tilde{R}(W,Z)V]. \tag{38}$$

The decompositions (36) and (38) provide an $\mathcal{O}(n)$-invariant subspace, characterized by invariant equations that are linear in $(\nabla_Z\mathrm{Ric})(V,W)$.

Therefore, the relation between $\nabla\mathrm{Ric}$ and the divergence of the Weyl conformal curvature tensor C can be given by the equation [31]

$$(\mathrm{div}\, C)(Z,V)W = \frac{n-3}{n-2}[\tilde{R}(Z,V)W - \tilde{R}(W,Z)V]. \tag{39}$$

From now on, we assume that $(M, g, \psi, \lambda, \alpha, \beta)$ is a GRYS space-time satisfying Einstein's field equations in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field.

According to Gray's decomposition, we may investigate the following different subspaces.

4.1. Trivial Subspace

This subspace is characterized by the condition $\nabla\mathrm{Ric} = 0$. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the trivial subspace, then as the scalar constant $b \neq 0$, by Eq. (31), we have $\nabla_Y\nabla\psi = 0$ for all $Y \in \chi(M)$, so the velocity field is parallel. By combining this result with (16), we obtain $\nabla T = 0$. Therefore, we can state that:

Theorem 4. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time satisfying Einstein's field equations in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the trivial subspace, then*

1. *the velocity field is parallel;*
2. *the energy-momentum tensor is covariantly constant.*

4.2. Subspace $\mathcal{I}$

This subspace is characterized by $\tilde{R}(Z,V)W = 0$, i.e.,

$$(\nabla_Z\mathrm{Ric})(V,W) = \alpha(Z)g(V,W) + \omega(V)g(Z,W) + \omega(W)g(V,Z). \tag{40}$$

By the relation (39) between the gradient of the Ricci tensor and the divergence of the Weyl conformal tensor, in this subspace, div $C = 0$ holds. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the subspace $\mathcal{I}$, then by (33) we have $\sigma + p = \dfrac{b}{\kappa} = \mathrm{const}$, i.e., it satisfies the $p = p(\sigma)$-like equation of state. Combining all these results with the Shepley–Taub classifications of space-times containing perfect fluids and having a vanishing conformal divergence [37], we conclude that:

Theorem 5. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the subspace $\mathcal{I}$, then*

1. *(M, g) is conformally flat;*
2. *(M, g) is endowed with the Robertson Walker metric;*
3. *the flow of this space-time is irrotational, and it has no shear.*

4.3. Subspace $\mathcal{A}$ (or Orthogonal Complement $\mathcal{I}'$)

This subspace is characterized by

$$(\nabla_Z \mathrm{Ric})(V, W) + (\nabla_V \mathrm{Ric})(Z, W) + (\nabla_W \mathrm{Ric})(Z, V) = 0. \tag{41}$$

That is, Ric is cyclic-parallel, and so the scalar curvature τ is constant. By taking the covariant derivative of the Einstein field equation (16), we get

$$(\nabla_Z \mathrm{Ric})(V, W) - \frac{d\tau(Z)}{2} g(V, W) = \kappa(\nabla_Z T)(V, W), \tag{42}$$

for all $Z, V, W \in \chi(M)$. Since τ is constant, (42) yields

$$(\nabla_Z T)(V, W) + (\nabla_V T)(Z, W) + (\nabla_W T)(Z, V) = 0, \tag{43}$$

for all $Z, V, W \in \chi(M)$. This means that T is Killing.

In [36], Sharma and Ghosh proved that for a perfect-fluid space-time admitting the Killing energy-momentum tensor, the expansion scalar and the shear tensor vanish, however, the vorticity tensor may or may not be zero. Thus we can state that:

Theorem 6. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the subspace $\mathcal{A}$, then*

1. *the energy-momentum tensor is Killing;*
2. *the flow of the space-time is expansion-free and shear-free, however, not necessarily vorticity-free.*

4.4. Subspaces $\mathcal{B}$ and $\mathcal{B}'$

These subspaces are characterized by a Codazzi-type Ricci tensor, i.e.,

$$(\nabla_Z \mathrm{Ric})(V, W) = (\nabla_V \mathrm{Ric})(Z, W), \tag{44}$$

for all $Z, V, W \in \chi(M)$, and so again τ is constant. A 4-dimensional Lorentzian manifold having a Codazzi-type Ricci tensor is called "Yang's pure space." In [16], Guilfoyle and Nolan proved that the necessary and sufficient condition for a 4-dimensional perfect-fluid space-time (M, g) with $\sigma + p \neq 0$ to be Yang's pure space is that (M, g) is a Robertson Walker space-time. Shaikh et al. [35] proved that in such a perfect fluid with Codazzi type energy momentum tensor, the fluid is shear-free and vorticity-free.

Petrov [34] made a classification that divided space-time into 6 types denoted as O, I, II, III, D, N. In [2], Barnes proved that if a perfect fluid space-time is shear-free, vorticity-free, and the velocity vector field of the fluid is hypersurface orthogonal, and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structures of space-time are of Petrov type I, D or O. According to all these discussions, the following result is obtained:

Theorem 7. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. If $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the subspaces $\mathcal{B}$ and $\mathcal{B}'$, then,*

1. *(M, g) is Yang's pure space, provided that the matter content is not vacuum,*
2. *(M, g) is a Robetson–Walker space-time,*
3. *(M, g) has vanishing shear and vorticity,*
4. *the possible local cosmological structures of (M, g) are of Petrov type I, D, or O.*

4.5. Subspace $\mathcal{I} \oplus \mathcal{A}$

This subspace is characterized by the extended cyclic condition

$$(\nabla_Z \mathrm{Ric})(V, W) + (\nabla_V \mathrm{Ric})(Z, W) + (\nabla_W \mathrm{Ric})(Z, V) = 2\frac{d\tau(Z)}{n+2} g(V, W) + 2\frac{d\tau(V)}{n+2} g(Z, W) + 2\frac{d\tau(W)}{n+2} g(Z, V). \tag{45}$$

But we already know that the scalar curvature τ is constant for the GRYS space-time $(M, g, \psi, \lambda, \alpha, \beta)$

satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. Therefore, this subspace directly reduces to subspace $\mathcal{A}$, and Theorem 6 is still valid if (M, g) belongs to the subspace $\mathcal{I} \oplus \mathcal{A}$.

4.6. Subspace $\mathcal{I} \oplus \mathcal{B}$

This subspace is characterized by the extended Codazzi condition

$$\nabla_Z\Big[\mathrm{Ric}(V,W) - \frac{\tau}{2(n-1)}g(V,W)\Big] = \nabla_V\Big[\,\mathrm{Ric}(Z,W) - \frac{\tau}{2(n-1)}g(Z,W)\Big], \quad (46)$$

which implies that the Weyl conformal curvature tensor C is divergence-free. Therefore, Theorem 5 is still valid if (M, g) belongs to the subspace $\mathcal{I} \oplus \mathcal{B}$.

4.7. Subspace $\mathcal{A} \oplus \mathcal{B}$

In this subspace, without any restrictions, the scalar curvature is constant. For a 4-dimensional perfect-fluid space-time (M, g), from (13), we have

$$\tau = 4a - b = \mathrm{const}. \quad (47)$$

From (17), (47) follows the equation of state

$$p = \frac{\sigma}{3} + C, \quad \text{where} \quad C \in \mathbb{R}. \quad (48)$$

Theorem 8. *Let $(M, g, \psi, \lambda, \alpha, \beta)$ be a GRYS space-time satisfying Einstein's field equations, in which the gradient of the potential function ψ is a unit-timelike torse-forming vector field. f $(M, g, \psi, \lambda, \alpha, \beta)$ belongs to the subspace $\mathcal{A} \oplus \mathcal{B}$, then the space-time satisfies the state equation* (48). *If, in particular, the scalar cuvature vanishes, then the space-time satisfies the state equation $\sigma = 3p$, i.e., the space-time represents the radiation era.*

4.8. Further Analysis

Before finishing our classification, let us now consider the case of $(M, g, \psi, \lambda, \alpha, \beta)$'s velocity vector field $\nabla\psi$ being a unit-timelike Killing vector. Then,

$$(\mathcal{L}_{\nabla\psi}g)(X,Y) = 0, \quad \forall X, Y \in \chi(M). \quad (49)$$

From the fundamental equation (1) on GRYS space-time, we have

$$\nabla_X\nabla\psi + \alpha QX = \left(\lambda - \frac{1}{2}\beta\tau\right)X. \quad (50)$$

By taking a covariant derivative of (50), we get for all $X, Y \in \chi(M)$:

$$\nabla_Y\nabla_X\nabla\psi + \alpha(\nabla_Y Q)X + \alpha Q(\nabla_Y X) = -\frac{1}{2}\beta dY(\tau)X + \left(\lambda - \frac{1}{2}\beta\tau\right)(\nabla_Y X). \quad (51)$$

If X and Y are replaced in Eq. (51), we get for all $X, Y \in \chi(M)$,

$$\nabla_X\nabla_Y\nabla\psi + \alpha(\nabla_X Q)Y + \alpha Q(\nabla_X Y) = -\frac{1}{2}\beta dX(\tau)Y + \Big(\lambda - \frac{1}{2}\beta\tau\Big)(\nabla_X Y). \quad (52)$$

Moreover, from (50) we have

$$\nabla_{[X,Y]}\nabla\psi + \alpha Q[X,Y] = \left(\lambda - \frac{1}{2}\beta\tau\right)[X,Y]. \quad (53)$$

From the last 3-equations (51)–(53), we obtain

$$R(X,Y)\nabla\psi = -\alpha\Big[(\nabla_X Q)Y - (\nabla_Y Q)X\Big] - \frac{1}{2}\beta[dX(\tau)Y - dY(\tau)X]. \quad (54)$$

By virtue of (18), Eq. (54) yields

$$\begin{aligned} R(X,Y)\nabla\psi = -\alpha\Big[&da(X)Y + db(X)\eta(Y)\nabla\psi \\ &+ b[(\nabla_X\eta)(Y)\nabla\psi + \eta(Y)(\nabla_X\nabla\psi)] \\ &- da(Y)X - db(Y)\eta(X)\nabla\psi \\ &- b[(\nabla_Y\eta)(X)\nabla\psi + \eta(X)(\nabla_Y\nabla\psi)]\Big] \\ &- \frac{1}{2}\beta[dX(\tau)Y - dY(\tau)X]. \end{aligned} \quad (55)$$

Contracting (55) over X, we get

$$\begin{aligned} \mathrm{Ric}(Y,\nabla\psi) &= \alpha(n-1)da(Y) - \alpha db(Y) \\ &- \alpha d(b)(\nabla\psi)\eta(Y) + b(\nabla_{\nabla\psi})(Y) + b\eta(Y)\mathrm{div}(\nabla\psi) \\ &+ \frac{1}{2}\beta(n-1)d\tau(Y). \end{aligned} \quad (56)$$

Also, from (11), for a perfect-fluid space-time with the velocity field $\nabla\psi$, we have

$$\mathrm{Ric}(Y,\nabla\psi) = ad\psi(Y) - b\eta(Y). \quad (57)$$

Putting $Y = \nabla\psi$ in the last two equations and comparing them, we get

$$3\alpha da(\nabla\psi) - b\mathrm{div}(\nabla\psi) + \frac{3}{2}\beta d(4a-b)(\nabla\psi) = -a + b. \quad (58)$$

In ([14], p. 89), Duggal and Sharma proved that for a 4-dimensional perfect-fluid space-time admitting a timelike Killing velocity field $\nabla\psi$, $\mathcal{L}_{\nabla\psi}\sigma = 0$ and $\mathcal{L}_{\nabla\psi}p = 0$. Thus by (17), $da(\nabla\psi) = db(\nabla\psi) = 0$, and also since $\nabla\psi$ is Killing, $\mathrm{div}(\nabla\psi) = 0$. Thus $a = b$ follows from (58). Finally, by combining this relation with (17), the following equation of state holds:

$$\sigma + 3p = 0 \quad (59)$$

As a result:

Theorem 9. *If a perfect-fluid space-time satisfying Einstein's field equations admits a $(M, g, \psi, \lambda, \alpha, \beta)$ GRYS whose velocity field is a unit-timelike Killing vector field $\nabla\psi$, then it represents a phantom era.*

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. L. J. Alias, A. Romero, and M. Sanchez, "Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson—Walker space-times," Gen. Rel. Grav. **27** (1), 71—84 (1995).
2. A. Barnes, "On shear free normal flows of a perfect fluid," Gen. Rel. Grav. **4** (2),105—129 (1973).
3. A. M. Blaga, "Almost η-Ricci solitons in (LCS)n-manifolds," Bull. Belg. Math. Soc. Simon Stevin. **25**(5), 641—653 (2018).
4. A. M. Blaga, "Solitons and geometrical structures in a perfect-fluid space-time," Rocky Mountain J. Math. **50** (1), 41—53 (2020).
5. A. M. Blaga and H. M. Taştan, "Some results on almost η-Ricci-Bourguignon solitons," J. Geom. Phys. **168**, 104316 (2021).
6. J.-P. Bourguignon, "Ricci curvature and Einstein metrics," in: *Global Differential Geometry and Global Analysis* (Berlin, 1979, Lecture Notes in Math. **838**, Springer, Berlin, 1981), pp. 42—63.
7. S. K. Chaubey, Y. J. Suh, and U. C. De, "Characterizations of the Lorentzian manifolds admitting a type of semi-symmetric metric connection," Anal. Math. Phys. **10** (61) (2020).
8. U. C. De and Y. J. Suh, "Some characterizations of Lorentzian manifolds," Int. J. Geom. Methods Mod. Phys. **16**, 1950016 (2019).
9. U. C. De and S. Shenawy, "Generalized quasi-Einstein GRW space-times," Int. J. Geom. Methods Mod. Phys. **16**, 1950124 (2019).
10. U. C. De, A. Sardar, and K. De, "Ricci—Yamabe solitons and 3-dimensional Riemannian manifolds," Turkish J. Math. **46** (3), 1078—1088 (2022).
11. K. De and U. C. De, "Ricci—Yamabe solitons in f(R)-gravity," Int. Electron. J. Geom. **16** (1), 334—342 (2023).
12. D. Dey, "Almost Kenmotsu metric as Ricci—Yamabe soliton," arXiv: 2005.02322.
13. D. Dey and P. Majhi, "Sasakian 3-Metric as a generalized Ricci—Yamabe soliton," Quaestiones Mathematicae 1—13 (2021).
14. K. L. Duggal and R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds* (Mathematics and Its Applications **487**, Springer, 1999).
15. A. Gray, "Einstein-like manifolds which are not Einstein," Geom. Dedicata **7**, 259—280 (1978).
16. B. S. Guilfoyle and B. C. Nolan, "Yang's gravitational theory," Gen. Rel Grav. **30** (3), 473—495 (1998).
17. S. Güler and S. Altay Demirbağ, "On Ricci symmetric generalized quasi Einstein space-times," Miskolc Math. Notes **16**, 859—868 (2015).
18. S. Güler and S. Altay Demirbağ, "A study of generalized quasi Einstein space-times with applications in general relativity," Int. J. Theor. Phys. **55**, 548—562 (2016).
19. S. Güler and M. Crasmareanu, "Ricci—Yamabe maps for Riemannian flows and their volume variation and volume entropy," Turkish J. Math. **43** (5), 2631—2641 (2019).
20. S. Güler and B. Ünal, "The existence of gradie nt yamabe solitons on space-times," Results in Math. **77** (5) (2022).
21. R. S. Hamilton, "Three-manifolds with positive Ricci curvature," J. Diff. Geom. **17** (2), 253—306 (1982).
22. R. S. Hamilton, "Lectures on Geometric Flows," unpublished (1989).
23. S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Spacetimes* (Cambridge University Press, Cambridge, 1973).
24. F. Karaca, "Gradient Ricci—Yamabe solitons on warped product manifolds," Filomat, **37** (7), 2199—2207 (2023).
25. M. Khatri, C. Zosangzuala and J. P. Singh, "Isometries on almost Ricci—Yamabe solitons," Arab. J. Math. **12**, 127—138 (2023).
26. S. Mallick and U. C. De, "On Pseudo Q-symmetric space-times," Anal. Math. Phys. **9**, 1333—1345 (2019).
27. C. A. Mantica, Y. J. Suh, and U. C. De, "A note on generalized Robertson—Walker space-times," Int. J. Geom. Meth. Mod. Phys. **13**, 1650079 (2016).
28. C. A. Mantica and L. G. Molinari, "On the Weyl and the Ricci tensors of generalized Robertson—Walker space-times," J. Math. Phys. **57** (10), 102502 (2016).
29. C. A. Mantica, L. G. Molinari, and U. C. De, "A condition for a perfect-fluid space-time to be a generalized Robertson—Walker space-time," J. Math. Phys. **57** (2) 022508 (2016), Erratum **57**, 049901 (2016).
30. C. A. Mantica and L. G. Molinari, "Generalized Robertson—Walker space-times—A survey," Int. J. Geom. Methods Mod. Phys. **14** (3), 1730001 (2017).
31. C. A. Mantica, L. G. Molinari, Y. J. Suh, and S. Shenawy, "Perfect-fluid, generalized Robertson—Walker space-time, and Gray's decomposition," J. Math. Phys. **60**, 052506 (2019).

32. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic Press, New York, 1983).
33. G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," arXiv: math/0211159.
34. A. Z. Petrov, *Einstein Spaces* (Pergamon Press, Oxford, 1949).
35. A. A. Shaikh, D. W. Yoon, and S. K. Hui, "On quasi-Einstein space-times," Tsukuba Journal Math. **33** (2), 305–326 (2009).
36. R. Sharma and A. Ghosh, "Perfect fluid space-times whose energy-momentum tensor is conformal Killing," J. Math. Phys. **51**, 022504 (2010).
37. L. C. Shepley and A. H. Taub, "Spacetimes containing perfect fluids and having a vanishing conformal divergence," Commun. Math. Phys. **5**, 237–256 (1967).
38. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2003).
39. K. Yano, "On torse forming direction in a Riemannian space," Proc. Imp. Acad. Tokyo **20**, 340–345 (1944).

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