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Pseudo-projective Tensor on Sequential Warped Products

Sinem Güler and Bülent Ünal

Abstract. The main objective of this paper is to study pseudo-projective tensor on sequential warped products and then to obtain necessary and sufficient conditions for a sequential warped product to be pseudo-projectively flat. Moreover, we also provide characterization of pseudo-projectively flat sequential generalized Robertson–Walker and pseudo-projectively flat sequential standard static spacetimes.

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1. Introduction

Let (M, g) be an n-dimensional $(n > 2)$ pseudo-Riemannian manifold. Then, the pseudo-projective curvature tensor P on M including the projective curvature tensor is defined by [24]

$$\begin{aligned}\mathrm{P}(X,Y)Z &= \alpha\mathrm{R}(X,Y)Z + \beta[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y] \\ &\quad -\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

where R is curvature tensor, Ric is the Ricci tensor, τ is the scalar curvature of (M, g) and $\alpha, \beta \in \mathbb{R} - \{0\}$.

If $\alpha = 1$ and $\beta = -\frac{1}{n-1}$ in (1.1), then the pseudo-projective curvature tensor takes the following form:

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y], \tag{1.2}$$

where $\mathcal{P}$ denotes the projective curvature tensor (see [19]). Thus, the projective curvature tensor is a particular case of the pseudo-projective curvature tensor. In a Riemannian manifold M, if there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in the

Euclidean space, such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor $\mathcal{P}$ vanishes. Moreover, (1.2) implies that M is projectively flat if and only if it is of constant curvature. Hence, the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. In [27], projective curvature tensors of a non-symmetric affine connection space are expressed as functions of the affine connection coefficients and Weyl projective tensor of the corresponding associated affine connection space. Moreover, projective flatness of non-symmetric affine connection spaces is analyzed. There are also two recent studies (see [4,9]) in the former one relations between pseudo-projectivity and warped products have been studied, and in the latter one, this structure was considered on Riemannian submersions equipped with quasi-conformal curvature. After that, pseudo-projectively Ricci semi-symmetric and pseudo-projectively flat (or conservative) semi-Riemannian manifolds are studied in [16]. In a recent paper [22], the projective curvature tensors are studied for special manifolds with non-symmetric linear connection. More explicitly, the authors obtained conditions for a nonsymmetric linear connection to obtain tensors having analogue expressions like the Weyl projective curvature tensor. In this paper, our aim is to extend these study to the sequential warped product manifolds and to obtain their relativistic applications.

The warped product of two Riemannian manifolds (B, g_B) and (F, g_F) as the product $B \times F$ equipped with the metric $g_B \oplus f^2 g_F$ and denoted by $B \times_f F$, where the smooth function $f \colon B \to (0, \infty)$ is called the warping function, [10]. Then, warped products of semi-Riemannian (not necessarily Lorentzian) manifolds and their Riemannian and Ricci curvature tensors were given in [20]. Then, it turned out that the standard spacetime models of the universe and many other fundamental examples of relativistic spacetimes as the solutions of Einstein's field equations can be expressed in terms of warped product manifolds. As a result of that, warped products play very important roles in differential geometry as well as in the theory of general relativity. Some of very well-known examples of this notion include Robertson–Walker spacetime, Einstein static spacetime, four dimensional Schwarzschild, and de Sitter solutions of Einstein field equations. In addition to that, generalizations of Robertson–Walker spacetime and Einstein static spacetime called as generalized Robertson–Walker spacetime (see [5,13]) and standard static spacetime (see [1–3]), respectively, have been extensively studied. Then, a new concept of warped product metric has been introduced in which its base or fiber or both also have the warped product structure and it is called sequential warped product [14,25]. In a recent article (see [23]), conformal metrics have been applied to the Einstein sequential warped product manifolds and are studied when the first warping function is equal to constant and the fiber manifold is flat. This approach allows to cover a wider variety of exact solutions of Einstein field equation, without complicating the calculations much, compared to the Einstein warped product manifolds with Ricci flat fiber. Moreover, in the very same study [23], the authors prove the existence of a family of Ric > 0 solutions and, among many others, that can be used to construct warped product spacetime models admitting positive curvature whose fiber is flat.

The formal definition of this notion is given as follows:

Definition 1. Let (M_i, g_i) be three semi-Riemannian manifolds of dimensions m_i, respectively, for any $i = 1, 2, 3$. Suppose that $f\colon M_1 \to (0, \infty)$ and $h\colon M_1 \times M_2 \to (0, \infty)$ are two smooth functions. Then, the sequential warped product is the product manifold $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ endowed with the metric

$$\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3. \tag{1.3}$$

Both f and g are called the warping functions. Also, both can not be constant at the same time. Otherwise, the manifold becomes simply a direct product.

Notation 1. *(i) Throughout the paper, we use Einstein's summation convention. All the objects will be assumed to be smooth and all the manifolds are connected.*

(ii) All objects having "bar" symbol represent the objects of the sequential warped product manifold and all objects having the indices or powers i denote the objects of the manifold (M_i, g_i), where $i = 1, 2, 3$.

(iii) The Riemann curvature tensor is defined by $\mathrm{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, the Ricci tensor is $\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \mathrm{R}(e_i, X, Y, e_i)$, and the scalar curvature $\tau = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$, where $\{e_i : i = 1, \ldots, n\}$ denotes an orthonormal basis on the manifold.

(iv) For any $X, Y \in \mathrm{T}(M)$, the Hessian of a smooth function ϕ is the second-order covariant differentiation defined by $\mathrm{H}^{\phi}(X, Y) = XY(\phi) - (\nabla_X Y)\phi = g(\nabla_X \mathrm{grad}\phi, Y)$.

(v) On a sequential warped product $\bar{M} = (M_1 \times_f M_2) \times_h M_3$, every vector field X can be decomposed as the sum

$$X = X_1 + X_2 + X_3, \quad \textit{where} \quad X_i \in \mathrm{T}(M_i), \quad i = 1, 2, 3.$$

In the next section, we have provided all the necessary curvature formulas related to the geometry of sequential warped products (for further details, see [17,18,21]). In Sect. 3, we derive expressions for the pseudo-projective tensor formulas of a sequential warped product to obtain our main results. Then, we consider pseudo-projectively flat sequential warped products and establish implications of this concept on the components of a sequential warped product. Finally, we apply the main results to characterize pseudo-projectively flat sequential generalized Robertson–Walker and pseudo-projectively flat sequential standard static spacetimes.

2. Preliminaries

In this section, we give the basic formulas for the Levi-Civita connection, Riemann, Ricci, and the scalar curvature of the sequential warped products that will be used throughout the study.

From now on, assume that $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ is a sequential warped product furnished with the metric $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and further assume that X_i, Y_i, Z_i are in $\mathrm{T}(M_i)$, for any $i = 1, 2, 3$. Then, we have the following:

Lemma 1. [14] *The components of the Levi–Civita connection on* $(\bar{M}, \bar{g})$ *are given by*

(1) $\bar{\nabla}_{X_1} Y_1 = \nabla^1_{X_1} Y_1$,
(2) $\bar{\nabla}_{X_1} X_2 = \bar{\nabla}_{X_2} X_1 = X_1(\ln f) X_2$,
(3) $\bar{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \mathrm{grad}^1 f$,
(4) $\bar{\nabla}_{X_3} X_1 = \bar{\nabla}_{X_1} X_3 = X_1(\ln h) X_3$,
(5) $\bar{\nabla}_{X_2} X_3 = \bar{\nabla}_{X_3} X_2 = X_2(\ln h) X_3$,
(6) $\bar{\nabla}_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) \mathrm{grad} h$.

Note that H^f_1 and $\Delta_1 f$ denote the Hessian and Laplacian of f on M_1, respectively, while H^h and Δh denote the Hessian and Laplacian of h on $\bar{M}$, respectively.

Lemma 2. [14] *The non-zero components of the Riemannian curvature of* $(\bar{M}, \bar{g})$ *are given by*

(1) $\bar{\mathrm{R}}(X_1, Y_1) Z_1 = \mathrm{R}_1(X_1, Y_1) Z_1$,
(2) $\bar{\mathrm{R}}(X_2, Y_2) Z_2 = \mathrm{R}_2(X_2, Y_2) Z_2 - ||\mathrm{grad}_1 f||^2 [g_2(Y_2, Z_2) X_2 - g_2(X_2, Z_2) Y_2]$,
(3) $\bar{\mathrm{R}}(X_1, Y_2) Z_1 = \frac{1}{f} \mathrm{H}^f_1(X_1, Z_1) Y_2$,
(4) $\bar{\mathrm{R}}(X_1, Y_2) Z_2 = -f g_2(Y_2, Z_2) \nabla^1_{X_1} \mathrm{grad}_1 f$,
(5) $\bar{\mathrm{R}}(X_i, Y_3) Z_j = \frac{1}{h} \mathrm{H}^h(X_i, Z_j) Y_3$, *for* $i, j = 1, 2$,
(6) $\bar{\mathrm{R}}(X_i, Y_3) Z_3 = -h g_3(Y_3, Z_3) \bar{\nabla}_{X_i} \mathrm{grad} h$, $i = 1, 2$,
(7) $\bar{\mathrm{R}}(X_3, Y_3) Z_3 = \mathrm{R}_3(X_3, Y_3) Z_3 - ||\mathrm{grad} h||^2 [g_3(Y_3, Z_3) X_3 - g_3(X_3, Z_3) Y_3]$.

Lemma 3. [14] *The non-zero components of the Ricci curvature of* $(\bar{M}, \bar{g})$ *are given by*

(1) $\bar{\mathrm{Ric}}(X_1, Y_1) = \mathrm{Ric}_1(X_1, Y_1) - \frac{m_2}{f} \mathrm{H}^f_1(X_1, Y_1) - \frac{m_3}{h} \mathrm{H}^h(X_1, Y_1)$,
(2) $\bar{\mathrm{Ric}}(X_2, Y_2) = \mathrm{Ric}_2(X_2, Y_2) - f^\sharp g_2(X_2, Y_2) - \frac{m_3}{h} \mathrm{H}^h(X_2, Y_2)$,
(3) $\bar{\mathrm{Ric}}(X_3, Y_3) = \mathrm{Ric}_3(X_3, Y_3) - h^\sharp g_3(X_3, Y_3)$,

where $f^\sharp = f \Delta_1 f + (m_2 - 1) ||\mathrm{grad}_1 f||^2$ *and* $h^\sharp = h \Delta h + (m_3 - 1) ||\mathrm{grad} h||^2$.

Lemma 4. [14] *The relation between the scalar curvature* $\bar{\tau}$ *of* $(\bar{M}, \bar{g})$ *and the scalar curvatures* τ_i *of* (M_i, g_i) *for any* $i = 1, 2, 3$, *is given by*

$$\begin{aligned} \bar{\tau} &= \tau_1 + \frac{r_2}{f^2} + \frac{r_3}{h^2} - \frac{2m_2}{f} \Delta_1 f - \frac{2m_3}{h} \Delta h \\ &\quad - \frac{m_2(m_2 - 1)}{f^2} ||\mathrm{grad}_1 f||^2 - \frac{m_3(m_3 - 1)}{h^2} ||\mathrm{grad} h||^2. \end{aligned}$$

In what follows, we use again Notations 1 to obtain the following:

Lemma 5. [17] *The Hessian tensor* H^φ *of* φ *on a sequential warped product* $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ *satisfies*

$$
\begin{aligned}
&(1)\ \mathrm{H}^\varphi(X_1, Y_1) = \mathrm{H}_1^\varphi(X_1, Y_1),\\
&(2)\ \mathrm{H}^\varphi(X_1, Y_2) = -X_1(\ln f)Y_2(\varphi),\\
&(3)\ \mathrm{H}^\varphi(X_1, Y_3) = -X_1(\ln h)Y_3(\varphi),\\
&(4)\ \mathrm{H}^\varphi(X_2, Y_2) = f\nabla\varphi(f)g_2(X_2, Y_2) + \mathrm{H}_2^\varphi(X_2, Y_2),\\
&(5)\ \mathrm{H}^\varphi(X_2, Y_3) = -X_2(\ln h)Y_3(\varphi),\\
&(6)\ \mathrm{H}^\varphi(X_3, Y_3) = h\nabla\varphi(h)g_3(X_3, Y_3) + \mathrm{H}_3^\varphi(X_3, Y_3).
\end{aligned}
$$

Also, note that for any function φ, the following relation holds:

$$
\frac{m}{\varphi}\mathrm{H}^\varphi = \mathrm{H}^{m\ln\varphi} + \frac{1}{m}\mathrm{d}(m\ln\varphi) \otimes \mathrm{d}(m\ln\varphi), \quad m \in \mathbb{R}. \tag{2.1}
$$

3. Pseudo-projective Tensor on Sequential Warped Products

In this section, some Ricci–Hessian class type manifolds are investigated on pseudo-projectively flat sequential warped products. If the pseudo-projective curvature tensor of (M, g) vanishes identically, then it is called pseudo-projectively flat.

Remark 1. Recall that every pseudo-projectively manifold is also Einstein.

An $n(> 2)$-dimensional smooth manifold (M^n, g) is said to be a generalized quasi-Einstein manifold in the sense of Catino [11] if there exist three smooth functions φ, α and λ on M, such that

$$
\mathrm{Ric} + \mathrm{H}^\varphi - \alpha \mathrm{d}\varphi \otimes \mathrm{d}\varphi = \lambda g, \tag{3.1}
$$

and it is denoted by $(M^n, g, \varphi, \alpha, \lambda)$. There are some different subclasses of this equation that define many important manifolds:

(i) If $\alpha = \frac{1}{m}$, for positive integer $0 < m < \infty$, then M is called an m-generalized quasi-Einstein manifold.

(ii) If $\alpha = 0$, then M is called a gradient almost Ricci soliton, [6] and it is denoted by (M, g, φ, λ) and φ is called the potential function.

(iii) Another particular case of (3.1) can be written as

$$
\mathrm{Ric} + \psi\mathrm{H}^\varphi = \lambda g. \tag{3.2}
$$

This structure is said to be a ψ-almost gradient Ricci soliton [30] and briefly denoted by $(M, g, \varphi, \psi, \lambda)$. For recent results for this class of solitons, we refer [29]. Note that, for every smooth function φ, the following relation can be verified:

$$
\mathrm{H}^{\ln\varphi} = \frac{1}{\varphi}\mathrm{H}^\varphi - \frac{1}{\varphi^2}\mathrm{d}\varphi \otimes \mathrm{d}\varphi. \tag{3.3}
$$

By defining a function $\phi = e^{\frac{-\varphi}{m}}$, we get $\frac{m}{\phi}\mathrm{H}^\phi = -\mathrm{H}^\varphi + \frac{1}{m}\mathrm{d}\varphi \otimes \mathrm{d}\varphi$. Thus, using (3.3), (3.1) can be written as

$$
\mathrm{Ric} - \frac{m}{\phi}\mathrm{H}^\phi = \lambda g. \tag{3.4}
$$

Hence, any m-generalized quasi-Einstein manifold is a $(\frac{-m}{\phi})$-almost gradient Ricci soliton.

(iv) More specificially, a Riemannian manifold (M^n, g) is called a conformal gradient soliton if there exists a non-constant smooth function f, called potential of the soliton, such that

$$\mathrm{H}^f = \varphi g, \tag{3.5}$$

for some function $\varphi : M^n \to \mathbb{R}$. Tashiro studied in [28] complete Riemannian manifolds admitting a vector field ∇f satisfying Eq. (3.5). Then, in [12], Cheeger and Colding gave the solutions of the equation (3.5) and they obtained the characterization of warped product manifolds. Also, Eq. (3.5) is the special case of gradient Ricci soliton structure, that has been studied extensively on the warped product manifold in [26]. For instance, in [7], Marco and Gouthier gave the classification of the warped products of constant sectional curvature satisfying the condition (3.5).

Theorem 1. *The non-zero components of the pseudo-projective curvature of a sequential warped product $(\bar{M}, \bar{g})$ of the form $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ equipped with $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ are as follows:*

(1)

$$\begin{aligned}\bar{\mathrm{P}}(X_1, Y_1)Z_1 = {} & \alpha \mathrm{R}_1(X_1, Y_1)Z_1 \\ & + \beta[\mathrm{Ric}_1(Y_1, Z_1)X_1 - \frac{m_2}{f}\mathrm{H}_1^f(Y_1, Z_1)X_1 - \frac{m_3}{h}\mathrm{H}^h(Y_1, Z_1)X_1] \\ & - \beta[\mathrm{Ric}_1(X_1, Z_1)Y_1 - \frac{m_2}{f}\mathrm{H}_1^f(X_1, Z_1)Y_1 - \frac{m_3}{h}\mathrm{H}^h(X_1, Z_1)Y_1] \\ & - \frac{\tau}{n}\big(\frac{\alpha}{n-1} + \beta\big)[g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1],\end{aligned}$$

(2)

$$\begin{aligned}\bar{\mathrm{P}}(X_2, Y_2)Z_2 = {} & \alpha \mathrm{R}_2(X_2, Y_2)Z_2 - \alpha\|\nabla_1 f\|^2[g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2] \\ & + \beta[\mathrm{Ric}_2(Y_2, Z_2)X_2 - f^\sharp g_2(Y_2, Z_2)X_2 - \frac{m_3}{h}\mathrm{H}^h(Y_2, Z_2)X_2] \\ & - \beta[\mathrm{Ric}_2(X_2, Z_2)Y_2 - f^\sharp g_2(X_2, Z_2)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_2, Z_2)Y_2] \\ & - \frac{\tau}{n}\big(\frac{\alpha}{n-1} + \beta\big)f^2[g_2(Y_2, Z_2)X_2 - g_1(X_2, Z_2)Y_2],\end{aligned}$$

(3)

$$\begin{aligned}\bar{\mathrm{P}}(X_1, Y_2)Z_1 = {} & \frac{\alpha}{f}\mathrm{H}_1^f(X_1, Z_1)Y_2 + \frac{\tau}{n}\big(\frac{\alpha}{n-1} + \beta\big)g_1(X_1, Z_1)Y_2 \\ & - \beta[\mathrm{Ric}_1(X_1, Z_1)Y_2 - \frac{m_2}{f}\mathrm{H}_1^f(X_1, Z_1)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_1, Z_1)Y_2],\end{aligned}$$

(4)

$$\begin{aligned}\bar{\mathrm{P}}(X_1, Y_2)Z_2 = {} & -\alpha f g_2(Y_2, Z_2)\nabla^1_{X_1}\nabla_1 f - \frac{\tau}{n}\big(\frac{\alpha}{n-1} + \beta\big)f^2 g_2(Y_2, Z_2)X_1 \\ & + \beta[\mathrm{Ric}_2(Y_2, Z_2)X_1 - f^\sharp g_2(Y_2, Z_2)X_1 - \frac{m_3}{h}\mathrm{H}^h(Y_2, Z_2)X_1],\end{aligned}$$

(5)

$$\begin{aligned}\bar{\mathrm{P}}(X_1,Y_3)Z_1 = {}& \frac{\alpha}{f}\mathrm{H}_1^f(X_1,Z_1)Y_3 + \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)g_1(X_1,Z_1)Y_3\\ &-\beta[\mathrm{Ric}_1(X_1,Z_1)Y_3 - \frac{m_2}{f}\mathrm{H}_1^f(X_1,Z_1)Y_3 - \frac{m_3}{h}\mathrm{H}^h(X_1,Z_1)Y_3],\end{aligned}$$

(6) $\bar{\mathrm{P}}(X_1,Y_3)Z_2 = \frac{\alpha}{h}\mathrm{H}^h(X_1,Z_2)Y_3,$

(7) $\bar{\mathrm{P}}(X_2,Y_3)Z_1 = \frac{\alpha}{h}\mathrm{H}^h(X_2,Z_1)Y_3,$

(8)

$$\begin{aligned}\bar{\mathrm{P}}(X_2,Y_3)Z_2 = {}& \frac{\alpha}{h}\mathrm{H}_1^f(X_2,Z_2)Y_3 + \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)f^2g_2(X_2,Z_2)Y_3\\ &-\beta[\mathrm{Ric}_2(X_2,Z_2)Y_3 - f^\sharp g_2(X_2,Z_2)Y_3 - \frac{m_3}{h}\mathrm{H}^h(X_2,Z_2)Y_3],\end{aligned}$$

(9)

$$\begin{aligned}\bar{\mathrm{P}}(X_1,Y_3)Z_3 = {}& -\alpha h g_3(Y_3,Z_3)\nabla_{X_1}\nabla h - \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)h^2g_3(Y_3,Z_3)X_1\\ &+\beta[\mathrm{Ric}_3(Y_3,Z_3)X_1 - h^\sharp g_3(Y_3,Z_3)X_1],\end{aligned}$$

(10)

$$\begin{aligned}\bar{\mathrm{P}}(X_2,Y_3)Z_3 = {}& -\alpha h g_3(Y_3,Z_3)\nabla_{X_2}\nabla h - \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)h^2g_3(Y_3,Z_3)X_2\\ &+\beta[\mathrm{Ric}_3(Y_3,Z_3)X_2 - h^\sharp g_3(Y_3,Z_3)X_2],\end{aligned}$$

(11)

$$\begin{aligned}\bar{\mathrm{P}}(X_3,Y_3)Z_3 = {}& \alpha\mathrm{R}_3(X_3,Y_3)Z_3 - \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)h^2[g_3(Y_3,Z_3)X_3 - g_3(X_3,Z_3)Y_3]\\ &+\beta[\mathrm{Ric}_3(Y_3,Z_3)X_3 - h^\sharp g_3(Y_3,Z_3)X_3]\\ &-\beta[\mathrm{Ric}_3(X_3,Z_3)Y_3 - h^\sharp g_3(X_3,Z_3)Y_3],\end{aligned}$$

where $X_i \in \mathrm{T}(M_i)$ *for any* $i = 1,2,3$ *and* $n = m_1+m_2+m_3$.

Now, we assume that the sequential warped product $(\bar{M},\bar{g})$ of the form $\bar{M} = (M_1 \times_f M_2)\times_h M_3$ equipped with $\bar{g} = (g_1 \oplus f^2g_2)\oplus h^2g_3$ is pseudo-projectively flat, i.e., $\bar{P} = 0$ on $(\bar{M},\bar{g})$.

First, notice that (6) and (7) of Theorem 1 imply that $\mathrm{H}^h(X_1,Z_2) = 0$ and $\mathrm{H}^h(X_2,Z_1) = 0$. Combining them with (2) of Lemma 5, we have either $X_1(\ln f) = 0$ for any $X_1 \in \mathfrak{X}(M_1)$ or $Z_2(h) = 0$ for any $Z_2 \in \mathfrak{X}(M_2)$. Thus, we conclude that either f is a constant or h depends only on M_1. Therefore, we may construct two sets $\mathcal{U} = \{p \in \bar{M} : f \text{ is a constant at p}\}$ and $\mathcal{V} = \{p \in \bar{M} : h \text{ depends only on } M_1 \text{ at p}\}$, such that $\bar{M} = \mathcal{U}\cup\mathcal{V}$.

After setting equations between (1) and (11) of Theorem 1, and contracting the resulting equation over Y_1 and Z_1, we obtain that

$$\begin{aligned}&(\alpha-\beta)\mathrm{Ric}_1(X_1,W_1) + \beta m_2\frac{1}{f}\mathrm{H}_1^f(X_1,W_1) + \beta m_3\frac{1}{h}\mathrm{H}^h(X_1,W_1)\\ &\quad = \Big[\frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)(m_1-1) - \beta\tau_1 + \beta m_2\frac{1}{f}\Delta_1 f + \beta m_3\frac{1}{h}\Delta_1 h\Big]g_1(X_1,W_1),\end{aligned} \tag{3.6}$$

for any $X_1, W_1 \in \chi\mathfrak{X}(\bar{M})$. Since $\bar{\mathrm{P}} = 0$, by (3) of Theorem 1, we have

$$\frac{\alpha}{f}\mathrm{H}_1^f(X_1, Z_1) - \beta\mathrm{Ric}_1(X_1, Z_1) + \beta\frac{m_2}{f}\mathrm{H}_1^f(X_1, Z_1) + \beta\frac{m_3}{h}\mathrm{H}^h(X_1, Z_1)$$
$$+\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)g_1(X_1, Z_1) = 0. \tag{3.7}$$

By combining Eqs. (3.6) and (3.7), also noting that $\alpha \neq 0$, we obtain that

$$\mathrm{Ric}_1 - \frac{1}{f}\mathrm{H}_1^f = \frac{\lambda_1}{\alpha}g_1, \tag{3.8}$$

where

$$\lambda_1 = \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)m_1 - \beta\tau_1 + \beta\frac{m_2}{f}\Delta_1 f + \beta\frac{m_3}{h}\Delta_1 h.$$

Thus, by (2.1), the above equation can be expressed as

$$\mathrm{Ric}_1 - \mathrm{H}_1^{\ln f} - \mathrm{d}(\ln f) \otimes \mathrm{d}(\ln f) = \frac{\lambda_1}{\alpha}g_1.$$

More explicitly, $(M_1, g_1, -\ln f, \lambda_1/\alpha)$ is a generalized quasi-Einstein manifold in the sense of Catino, where the potential function is determined in terms of the warping function.

Again since $\bar{\mathrm{P}} = 0$, the item (2) of Theorem 1 becomes

$$\begin{aligned}-\alpha\mathrm{R}_2(X_2, Y_2)Z_2 = {} & -\alpha\|\nabla_1 f\|^2[g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2] \\ & +\beta[\mathrm{Ric}_2(Y_2, Z_2)X_2 - f^\sharp g_2(Y_2, Z_2)X_2 - \frac{m_3}{h}\mathrm{H}^h(Y_2, Z_2)X_2] \\ & -\beta[\mathrm{Ric}_2(X_2, Z_2)Y_2 - f^\sharp g_2(X_2, Z_2)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_2, Z_2)Y_2] \\ & -\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)f^2[g_2(Y_2, Z_2)X_2 - g_1(X_2, Z_2)Y_2].\end{aligned}$$

Contracting the last equation over Y_2 and Z_2 and then using the identity (4) of Lemma 5, we obtain

$$(\alpha - \beta)\mathrm{Ric}_2(X_2, W_2) + \frac{\mu}{h}\mathrm{H}_2^h(X_2, W_2) = \lambda g_2(X_2, W_2), \tag{3.9}$$

where

$$\begin{aligned}\lambda_2 = {} & -\beta\tau_2 + \beta m_2 f^\sharp + \beta m_3(m_2 f\nabla h(f) + \Delta h)\frac{1}{h} - \beta f^\sharp \\ & -\beta m_3\frac{f}{h}\nabla h(f) + (m_2 - 1)\Big(\alpha\|\nabla_1 f\|^2 + \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)f^2\Big)\end{aligned}$$

and $\mu(\alpha - \beta) = \beta m_3$.

If $\alpha \neq \beta$, then by (3.9) and (2.1), we can state

$$\mathrm{Ric}_2 + \mathrm{H}^{\mu\ln h} + \frac{1}{\mu}\mathrm{d}(\mu\ln h) \otimes \mathrm{d}(\mu\ln h) = \lambda_2 g_2.$$

More explicitly, $(M_2, g_2, \mu\ln h, \lambda_2)$ is a generalized quasi-Einstein manifold in the sense of Catino, where the potential function is determined in terms of the warping function. If $\alpha = \beta$, then (3.9) reduces to the relation of conformal gradient soliton.

Finally, since $\bar{\mathrm{P}} = 0$, the item (11) of Theorem 1 implies

$$\begin{aligned}-\alpha \mathrm{R}_3(X_3, Y_3)Z_3 &= \beta[\mathrm{Ric}_3(Y_3, Z_3)X_3 - h^{\sharp} g_3(Y_3, Z_3)X_3] \\ &\quad -\beta[\mathrm{Ric}_3(X_3, Z_3)Y_3 - h^{\sharp} g_3(X_3, Z_3)Y_3] \\ &\quad -\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) h^2[g_3(Y_3, Z_3)X_3 - g_3(X_3, Z_3)Y_3].\end{aligned}$$

After contracting the last equation over Y_3 and Z_3, we have

$$(\alpha - \beta)\mathrm{Ric}_3 = \left[\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) h^2(m_3 - 1) - \beta\tau_3 + \beta h^{\sharp}(m_3 - 1)\right] g_3.$$

That is, (M_3, g_3) is Einstein.

All the above findings can be summarized as follows:

Theorem 2. *If a sequential warped product $(\bar{M}, \bar{g})$ of the form $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ equipped with $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ is pseudo-projectively flat, then $\bar{M} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{U} = \{p \in \bar{M} : f \text{ is a constant at } p\}$ and $\mathcal{V} = \{p \in \bar{M} : h \text{ depends only on } M_1 \text{ at } p\}$. In this case, the following three cases can occur:*

1. *For any $p \in \mathcal{U}$,*
    - *(a) (M_1, g_1) is an Einstein manifold,*
    - *(b) $(M_2, g_2, \mu \ln h, \lambda_2)$ is a generalized quasi-Einstein manifold, provided that $\alpha \neq \beta$. Otherwise, it is conformal gradient soliton.*
    - *(c) (M_3, g_3) is Einstein, provided that $\alpha \neq \beta$.*
2. *For any $p \in \mathcal{V}$,*
    - *(a) $(M_1, g_1, -\ln f, \lambda_1/\alpha)$ is a generalized quasi-Einstein manifold,*
    - *(b) (M_2, g_2) is an Einstein manifold,*
    - *(c) (M_3, g_3) is Einstein, provided that $\alpha \neq \beta$.*
3. *For any $p \in \mathcal{U} \cap \mathcal{V}$, (M_i, g_i), $(i = 1, 2, 3)$ are Einstein manifolds.*

Remark 2. If $\alpha = 1$ and $\beta = -1/(n-1)$, then $\bar{\mathrm{P}}$ is reduced to the projective curvature tensor. Hence, Theorem 2 holds also for projectively flat sequential warped products.

Note that if the Ricci tensor of the manifold satisfies the condition

$$(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z), \tag{3.10}$$

which means that the manifold has the Codazzi type Ricci tensor, [8,15]. Hence, we can state the following:

Corollary 1. *Let $(\bar{M}, \bar{g})$ be a sequential warped product of the form $\bar{M} = (M_1 \times_f M_2) \times_h M_3$. Assume that $(\bar{M}, \bar{g})$ is pseudo-projectively conservative sequential warped product. If $\alpha + \beta \neq 0$, and $(\bar{M}, \bar{g})$ is of constant scalar curvature, then the Ricci tensor of $(\bar{M}, \bar{g})$ is of Codazzi type.*

Proof. Note that if $\mathrm{div}(\bar{\mathrm{P}}) = 0$, then $\mathrm{div}\bar{\mathrm{C}} = 0$ or $\alpha + \beta \neq 0$, where $\bar{\mathrm{C}}$ denotes for Weyl tensor. □

4. Applications

4.1. Sequential Generalized Robertson–Walker Spacetimes

A sequential generalized Robertson–Walker spacetime is a sequential warped product of the form $\bar{M} = (I \times_f M_2) \times_h M_3$ endowed with the metric $\bar{g} = (-\mathrm{d}t^2 \oplus f^2 g_2) \oplus h^2 g_3$.

Using Theorem 1, one can easily state the following:

Theorem 3. *The non-zero components of the pseudo-projective curvature of a sequential generalized Robertson–Walker spacetime* $(\bar{M}, \bar{g})$ *of the form* $\bar{M} = (I \times_f M_2) \times_h M_3$ *endowed with the metric* $\bar{g} = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$ *are as follows:*

(1)

$$\begin{aligned}\bar{\mathrm{P}}(X_2, Y_2)Z_2 = {} & \alpha \mathrm{R}_2(X_2, Y_2)Z_2 + \alpha(\dot{f})^2[g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2] \\ & + \beta[\mathrm{Ric}_2(Y_2, Z_2)X_2 - f^{\sharp} g_2(Y_2, Z_2)X_2 - \frac{m_3}{h}\mathrm{H}^h(Y_2, Z_2)X_2] \\ & - \beta[\mathrm{Ric}_2(X_2, Z_2)Y_2 - f^{\sharp} g_2(X_2, Z_2)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_2, Z_2)Y_2] \\ & - \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) f^2[g_2(Y_2, Z_2)X_2 - g_1(X_2, Z_2)Y_2],\end{aligned}$$

(2)

$$\bar{\mathrm{P}}(\partial_t, Y_2)\partial_t = -\alpha\frac{\ddot{f}}{f}Y_2 - \beta\big[m_2\frac{\ddot{f}}{f} + m_3\frac{1}{h}\frac{\partial^2 h}{\partial t^2}\big]Y_2 - \frac{\tau}{n}\big(\frac{\alpha}{n-1} + \beta\big)Y_2,$$

(3)

$$\begin{aligned}\bar{\mathrm{P}}(\partial_t, Y_2)Z_2 = {} & -\alpha f\ddot{f} g_2(Y_2, Z_2)\partial_t - \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) f^2 g_2(Y_2, Z_2)\partial_t \\ & + \beta[\mathrm{Ric}_2(Y_2, Z_2) - f^{\sharp} g_2(Y_2, Z_2) - \frac{m_3}{h}\mathrm{H}^h(Y_2, Z_2)]\partial_t,\end{aligned}$$

(4)

$$\bar{\mathrm{P}}(\partial_t, Y_3)\partial_t = -\frac{\alpha}{h}\frac{\partial^2 h}{\partial t^2}Y_3 - \beta\big[m_2\frac{\ddot{f}}{f} - m_3\frac{1}{h}\frac{\partial^2 h}{\partial t^2}\big]Y_3 - \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)Y_3,$$

(5)

$$\begin{aligned}\bar{\mathrm{P}}(X_2, Y_3)Z_2 = {} & \frac{\alpha}{h}\mathrm{H}^f_1(X_2, Z_2)Y_3 + \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) f^2 g_2(X_2, Z_2)Y_3 \\ & - \beta[\mathrm{Ric}_2(X_2, Z_2)Y_3 - f^{\sharp} g_2(X_2, Z_2)Y_3 - \frac{m_3}{h}\mathrm{H}^h(X_2, Z_2)Y_3],\end{aligned}$$

(6)

$$\begin{aligned}\bar{\mathrm{P}}(\partial_t, Y_3)Z_3 = {} & -\alpha h g_3(Y_3, Z_3)\nabla_{\partial_t}\nabla h - \frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right) h^2 g_3(Y_3, Z_3)\partial_t \\ & + \beta[\mathrm{Ric}_3(Y_3, Z_3) - h^{\sharp} g_3(Y_3, Z_3)]\partial_t,\end{aligned}$$

(7)

$$\bar{\mathrm{P}}(X_2,Y_3)Z_3 = -\alpha h g_3(Y_3,Z_3)\nabla_{X_2}\nabla h - \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right)h^2 g_3(Y_3,Z_3)X_2 \\ +\beta[\mathrm{Ric}_3(Y_3,Z_3)X_2 - h^\sharp g_3(Y_3,Z_3)X_2],$$

(8)

$$\bar{\mathrm{P}}(X_3,Y_3)Z_3 = \alpha \mathrm{R}_3(X_3,Y_3)Z_3 - \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right)h^2[g_3(Y_3,Z_3)X_3 - g_3(X_3,Z_3)Y_3] \\ +\beta[\mathrm{Ric}_3(Y_3,Z_3)X_3 - h^\sharp g_3(Y_3,Z_3)X_3] \\ -\beta[\mathrm{Ric}_3(X_3,Z_3)Y_3 - h^\sharp g_3(X_3,Z_3)Y_3],$$

where $X_i \in \mathrm{T}(M_i)$ *for any* $i = 1,2,3$ *and* $n = 1 + m_2 + m_3$.

Now, we will consider implications of the case given by $\bar{\mathrm{P}} = 0$.

Equations (2) and (4) of Theorem 3 take the following forms, respectively:

$$-\alpha\frac{\ddot{f}}{f} - \beta\big[m_2\frac{\ddot{f}}{f} + m_3\frac{1}{h}\frac{\partial^2 h}{\partial t^2}\big] - \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big) = 0,$$

$$-\frac{\alpha}{h}\frac{\partial^2 h}{\partial t^2} - \beta\big[m_2\frac{\ddot{f}}{f} + m_3\frac{1}{h}\frac{\partial^2 h}{\partial t^2}\big] - \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big) = 0.$$

In view of the last two equations, one can easily obtain

$$\frac{\ddot{f}}{f} = \frac{1}{h}\frac{\partial^2 h}{\partial t^2}.$$

Similarly, Eqs. (3) and (5) of Theorem 3 become the followings, respectively:

$$-\alpha f\ddot{f}g_2 + \beta[\mathrm{Ric}_2 - f^\sharp g_2 - \frac{m_3}{h}\mathrm{H}^h] - \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right)f^2 g_2 = 0,$$

$$\frac{\alpha}{h}\mathrm{H}^f_1 - \beta[\mathrm{Ric}_2 - f^\sharp g_2 - \frac{m_3}{h}\mathrm{H}^h] + \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right)f^2 g_2 = 0.$$

Hence, from the last two equations

$$\frac{1}{h}\mathrm{H}^h = f\ddot{f}g_2 \quad \text{on} \quad M_2.$$

The last equation implies that

$$\mathrm{H}^h_2 = \big[hf\ddot{f} - f\nabla h(f)\big]g_2.$$

That is, (M_2, g_2) is a conformal gradient soliton. Also, by contracting the previous equation, we get

$$\Delta_2 h = m_2\big[hf\ddot{f} - f\nabla h(f)\big].$$

Moreover, from Eqs. (6) and (7) of Theorem 3, the warping function $h \in \mathcal{C}^\infty(M_1 \times M_2)$ satisfies and we have

$$\nabla_{\partial t}\nabla h = \nabla_{X_2}\nabla h.$$

Again setting $\bar{\mathrm{P}} = 0$ in Eq. (3) of Theorem 3 and then combining the resulting equation with (1) of Theorem 3 (note that $\alpha \neq 0$) result in

$$\mathrm{R}_2(X_2, Y_2)Z_2 = -\big(f\ddot{f} + (\dot{f})^2\big)[g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2].$$

Hence, (M_2, g_2) is of constant sectional curvature with $k_2 = -\big(f\ddot{f} + (\dot{f})^2\big)$ and thus

$$\mathrm{Ric}_2(X_2, Y_2) = -\big(f\ddot{f} + (\dot{f})^2\big)(m_2 - 1)g_2(X_2, Y_2).$$

That is, (M_2, g_2) is Einstein. Similarly, (M_3, g_3) is Einstein, as well.

Therefore, we can state the following result.

Theorem 4. *Let* $(\bar{M}, \bar{g})$ *be a sequential generalized Robertson–Walker spacetime of the form* $\bar{M} = (I \times_f M_2) \times_h M_3$ *endowed with the metric* $\bar{g} = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$. *If* $(\bar{M}, \bar{g})$ *is pseudo-projectively flat, then*

(1) the warping functions f *and* h *satisfy*

$$\frac{\ddot{f}}{f} = \frac{1}{h}\frac{\partial^2 h}{\partial t^2},$$

(2) (M_2, g_2) *is a conformal gradient soliton and both* (M_2, g_2) *and* (M_3, g_3) *are Einstein.*

If $f\ddot{f} + (\dot{f})^2 = 0$, then $(f\dot{f})$ =constant, which yields $f^2(t) = at + b$, for some constant a, b. Thus, we can state the following:

Corollary 2. *Let* $(\bar{M}, \bar{g})$ *be a sequential generalized Robertson–Walker spacetime of the form* $\bar{M} = (I \times_f M_2) \times_h M_3$ *endowed with the metric* $\bar{g} = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$. *If* $(\bar{M}, \bar{g})$ *is pseudo-projectively flat and* $f^2(t) = at + b$, *for some constant* a, b, *then* (M_2, g_2) *is flat.*

4.2. Sequential Standard Static Spacetimes

A sequential standard static spacetime is a sequential warped product of the form $\bar{M} = (M_1 \times_f M_2) \times_h I$ endowed with the metric $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-\mathrm{d}t^2)$. As an application of Theorem 1 and Theorem 2, the following results are directly obtained. Therefore, we omit their proofs.

Theorem 5. *The non-zero components of the pseudo-projective curvature of a sequential standard static spacetime* $(\bar{M}, \bar{g})$ *of the form* $\bar{M} = (M_1 \times_f M_2) \times_h I$ *equipped with* $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$ *are as follows:*

(1)

$$\begin{aligned}\bar{\mathrm{P}}(X_1, Y_1)Z_1 = \alpha \mathrm{R}_1(X_1, Y_1)Z_1 &+\beta[\mathrm{Ric}_1(Y_1, Z_1)X_1 - \frac{m_2}{f}\mathrm{H}_1^f(Y_1, Z_1)X_1 - \frac{m_3}{h}\mathrm{H}^h(Y_1, Z_1)X_1] \\ &-\beta[\mathrm{Ric}_1(X_1, Z_1)Y_1 - \frac{m_2}{f}\mathrm{H}_1^f(X_1, Z_1)Y_1 - \frac{m_3}{h}\mathrm{H}^h(X_1, Z_1)Y_1] \\ &-\frac{\tau}{n}\left(\frac{\alpha}{n-1} + \beta\right)[g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1],\end{aligned}$$

(2)

$$\begin{aligned}\bar{\mathrm{P}}(X_2,Y_2)Z_2 = {} & \alpha\mathrm{R}_2(X_2,Y_2)Z_2 - \alpha\|\nabla_1 f\|^2[g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2] \\ & +\beta[\mathrm{Ric}_2(Y_2,Z_2)X_2 - f^\sharp g_2(Y_2,Z_2)X_2 - \frac{m_3}{h}\mathrm{H}^h(Y_2,Z_2)X_2] \\ & -\beta[\mathrm{Ric}_2(X_2,Z_2)Y_2 - f^\sharp g_2(X_2,Z_2)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_2,Z_2)Y_2] \\ & -\frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right) f^2[g_2(Y_2,Z_2)X_2 - g_1(X_2,Z_2)Y_2],\end{aligned}$$

(3)

$$\begin{aligned}\bar{\mathrm{P}}(X_1,Y_2)Z_1 = {} & \frac{\alpha}{f}\mathrm{H}_1^f(X_1,Z_1)Y_2 + \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right) g_1(X_1,Z_1)Y_2 \\ & -\beta[\mathrm{Ric}_1(X_1,Z_1)Y_2 - \frac{m_2}{f}\mathrm{H}_1^f(X_1,Z_1)Y_2 - \frac{m_3}{h}\mathrm{H}^h(X_1,Z_1)Y_2],\end{aligned}$$

(4)

$$\begin{aligned}\bar{\mathrm{P}}(X_1,Y_2)Z_2 = {} & -\alpha f g_2(Y_2,Z_2)\nabla^1_{X_1}\nabla_1 f - \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right) f^2 g_2(Y_2,Z_2)X_1 \\ & +\beta[\mathrm{Ric}_2(Y_2,Z_2)X_1 - f^\sharp g_2(Y_2,Z_2)X_1 - \frac{m_3}{h}\mathrm{H}^h(Y_2,Z_2)X_1],\end{aligned}$$

(5)

$$\begin{aligned}\bar{\mathrm{P}}(X_1,\partial_t)Z_1 = {} & \frac{\alpha}{f}\mathrm{H}_1^f(X_1,Z_1)\partial_t + \frac{\tau}{n}\left(\frac{\alpha}{n-1}+\beta\right) g_1(X_1,Z_1)\partial_t \\ & -\beta[\mathrm{Ric}_1(X_1,Z_1)\partial_t - \frac{m_2}{f}\mathrm{H}_1^f(X_1,Z_1)\partial_t - \frac{m_3}{h}\mathrm{H}^h(X_1,Z_1)\partial_t],\end{aligned}$$

(6) $\bar{\mathrm{P}}(X_1,\partial_t)Z_2 = \dfrac{\alpha}{h}\mathrm{H}^h(X_1,Z_2)\partial_t,$

(7) $\bar{\mathrm{P}}(X_2,\partial_t)Z_1 = \dfrac{\alpha}{h}\mathrm{H}^h(X_2,Z_1)\partial_t,$

(8)

$$\begin{aligned}\bar{\mathrm{P}}(X_2,\partial_t)Z_2 = {} & \frac{\alpha}{h}\mathrm{H}_1^f(X_2,Z_2)\partial_t + \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big) f^2 g_2(X_2,Z_2)\partial_t \\ & -\beta[\mathrm{Ric}_2(X_2,Z_2)\partial_t - f^\sharp g_2(X_2,Z_2)\partial_t - \frac{m_3}{h}\mathrm{H}^h(X_2,Z_2)\partial_t],\end{aligned}$$

(9)

$$\bar{\mathrm{P}}(X_1,\partial_t)\partial_t = \alpha h\nabla_{X_1}\nabla h + \beta h^\sharp X_1 + \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)h^2 X_1,$$

(10)

$$\bar{\mathrm{P}}(X_2,\partial_t)\partial_t = \alpha h\nabla_{X_2}\nabla h + \beta h^\sharp X_2 + \frac{\tau}{n}\big(\frac{\alpha}{n-1}+\beta\big)h^2 X_2,$$

where $X_i \in \mathrm{T}(M_i)$ *for any* $i = 1, 2, 3$ *and* $n = 1 + m_2 + m_3$.

Therefore, as a direct application of Theorem 2 and Theorem 5, we may conclude that:

Theorem 6. *Let* $(\bar{M}, \bar{g})$ *be a sequential standard static spacetime of the form* $\bar{M} = (M_1 \times_f M_2) \times_h I$ *endowed with the metric* $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$. *If* $(\bar{M}, \bar{g})$ *is pseudo-projectively flat, then* $\bar{M} = \mathcal{U} \cup \mathcal{V}$, *where* $\mathcal{U} = \{p \in \bar{M} : f \text{ is a constant at } p\}$ *and* $\mathcal{V} = \{p \in \bar{M} : h \text{ depends only on } M_1 \text{ at } p\}$. *In this case, the following three cases can occur:*

- *(1) For any* $p \in \mathcal{U}$,
    - *(a)* (M_1, g_1) *is an Einstein manifold,*
    - *(b)* $(M_2, g_2, \mu \ln h, \lambda_2)$ *is a generalized quasi-Einstein manifold, provided that* $\alpha \neq \beta$. *Otherwise, it is conformal gradient soliton.*
- *(2) For any* $p \in \mathcal{V}$,
    - *(a)* $(M_1, g_1, -\ln f, \lambda_1/\alpha)$ *is a generalized quasi-Einstein manifold,*
    - *(b)* (M_2, g_2) *is an Einstein manifold,*
- *(3) For any* $p \in \mathcal{U} \cap \mathcal{V}$, (M_i, g_i), *(*$i = 1, 2$*) are Einstein manifolds.*

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Declarations

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Sinem Güler
Department of Industrial Engineering
Istanbul Sabahattin Zaim University
Halkali
34303 Istanbul
Turkey
e-mail: `sinem.guler@izu.edu.tr`

Bülent Ünal
Department of Mathematics
Bilkent University
Bilkent
06800 Ankara
Turkey
e-mail: `bulentunal@mail.com`

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