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Research Article

ϕ -δ-Primary Hyperideals in Krasner Hyperrings

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In this paper, we study commutative Krasner hyperrings with nonzero identity. ϕ-prime, ϕ-primary and ϕ-δ-primary hyperideals are introduced. The concept of δ-primary hyperideals is extended to ϕ-δ-primary hyperideals. Some characterizations of hyperideals are provided to classify them. The relation between ϕ-δ-primary hyperideals and other hyperideals is discussed.

1. Introduction

In commutative ring theory, prime and primary ideals have a significant place. The importance of prime ideals encourages researchers to expand these concepts and find applications. Many different types of generalizations have been investigated by several authors, some of them [1–5]. Prime and primary ideals are generalized to ϕ -prime and ϕ-primary ideals. Let $(\mathfrak{R}, +, .)$ be a commutative ring with nonzero identity. Denote the set of all ideals of $\mathfrak{R}$ by $L(\mathfrak{R})$ (proper ideals of $\mathfrak{R}$ by $L^*(\mathfrak{R})$. Let ϕ be a function such that $\phi\colon L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$. Let N be a proper ideal of $\mathfrak{R}$. N is called a ϕ-prime ideal [1], if $ab \in N - \phi(N)$, then either $a \in N$ or $b \in N$ for some $a, b \in \mathfrak{R}$. By the way, N is called a ϕ-primary ideal when, if $ab \in N - \phi(N)$, then $a \in N$ or $b^k \in N$ for some $a, b \in \mathfrak{R}$, $k \in \mathbb{N}$ [3, 4]. The image of ideal, $\phi(N)$, can be equal 0, $\varnothing, N, N^2, N^n, N^w$ (w denotes the intersection of ideals of N_i). A proper ideal N of $\mathfrak{R}$ is called weakly prime (primary) ideal respectively in [6] ([7]) if $0 \neq ab \in N$, for some $a, b \in \mathfrak{R}$, then $a \in N$ or $b \in N$ ($b^k \in N$ for some $k \in \mathbb{N}$). Anderson generalized it in [1], where N is weakly ϕ-prime ideal when $\phi(N) = 0$. Zhao [8] introduced δ-primary ideal as an expansion of an ideal, $\delta\colon L(\mathfrak{R}) \longrightarrow L(\mathfrak{R})$ is a function that meets the following requirements: $i) N \subseteq \delta(N)$, for all ideals N of $\mathfrak{R}$, $ii)$ If $N \subseteq M$, where N and M are ideals of $\mathfrak{R}$, then $\delta(N) \subseteq \delta(M)$, $iii) \delta(K \cap L) = \delta(K) \cap \delta(L)$ for all ideals K, L of $\mathfrak{R}$. Entire of the δ ideal expansions provides the property $\delta^2 = \delta$, which is $\delta(\delta(N)) = \delta(N)$ for all ideal N of $\mathfrak{R}$ [8]. A. Jaber chose ϕ such a reduction function in [9], which satisfies the following requirements: $i) \phi(N) \subseteq N$, for all ideals N of $\mathfrak{R}$, $ii)$ If $N \subseteq M$, where N and M are ideals of $\mathfrak{R}$, then $\phi(N) \subseteq \phi(M)$. He obtained generalization of ϕ-δ-primary ideal by combining these two concepts. Let N be an ideal of $\mathfrak{R}$, δ be an ideal expansion and ϕ be an ideal reduction [9]. N is called ϕ- δ-primary if $ab \in N - \phi(N)$, then either $a \in N$ or $b \in \delta(N)$, for all $a, b \in \mathfrak{R}$. Some results on $\varphi - \delta$-primary ideals can be found in [10, 11].

The theory of hyperstructures was innovated by Marty in 1934 [12]. He defined hypergroupoid $(G, ^\circ)$ for $G \neq \varnothing$, $P^*(G)$ represents family of nonempty subsets of G and $^\circ\colon \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{P}^*(\mathrm{G})$ is a binary hyperoperation. Let $(G, ^\circ)$ be a hypergroupoid. G is a semihypergroup, if $\forall a, b, c \in G$, $a^\circ(b^\circ c) = (a^\circ b)^\circ c$, which means $\cup_{u \in a^\circ b} u^\circ c = \cup_{v \in b^\circ c} a^\circ v$. If

$\forall a \in G$, there exists $e \in G$ such that $a \in (e^\circ a) \cap (a^\circ e)$ in another phrase $\{a\} \subseteq (e^\circ a) \cap (a^\circ e)$, then e is called identity element. An identity element e is called a scalar identity if $\{a\} = (e^\circ a) \cap (a^\circ e)$, for all $a \in G$. Let $(G,^\circ)$ be a semihypergroup. For $\forall a \in G$, if $a^\circ G = G^\circ a = G$, then $(G,^\circ)$ is called hypergroup. Let $(G,^\circ)$ be a hypergroup and $\varnothing \neq K$ be a subset of G. If $a^\circ K = K^\circ a = K$, for $\forall a \in K$, then $(K,^\circ)$ is called subhypergroup of $(G,^\circ)$ where $\circ$ is a binary hyperoperation on G. Let $(G,^\circ)$ be a hypergroup. If $a^\circ b = b^\circ a$, for $\forall a, b \in G$, then $(G,^\circ)$ is commutative hypergroup [12]. Mittas pioneered the theory of canonical hypergroups in [13–15]: Let $\mathfrak{R} \neq \varnothing$. $(\mathfrak{R}, +)$ is called a canonical hypergroup (+ is a hyperoperation) if the following axioms are satisfied: $i) a + (b + c) = (a + b) + c$, for $a, b, c \in \mathfrak{R}$; $ii)$ $a + b = b + a$, for $a, b \in \mathfrak{R}$; $iii)$ $\exists 0 \in \mathfrak{R}$ such that $a + 0 = \{a\}$, for any $a \in \mathfrak{R}$; $iv)$ for any $a \in \mathfrak{R}$, there exists a unique element $a\prime \in \mathfrak{R}$, such that $0 \in a + a\prime$ ($a\prime$ is called as the opposite of a and it is denoted by $-a$); $v)$ $c \in a + b$ implies that $b \in -a + c$ and $a \in c - b$, which means $(\mathfrak{R}, +)$ is reversible. Hyperrings and hyperfields were introduced by Krasner in [16] using the canonical hypergroups. $(\mathfrak{R}, +, \cdot)$ is called Krasner hyperring if the following statements hold: $i)$ $(\mathfrak{R}, +)$ is a canonical hypergroup; $ii)$ $(\mathfrak{R}, \cdot)$ is a semigroup having 0 as $a \cdot 0 = 0 \cdot a = 0$, for all $a \in \mathfrak{R}$; $iii)$ $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, for all $a, b, c \in \mathfrak{R}$. Also, Ameri and Norouzi [17] studied general commutative hyperrings. Corsini and Leoreanu [18] presented some of the most important results on hyperrings and illustrated some of the most recent and interesting applications, that is, those to geometry, graphs and hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras and C-algebras, artificial intelligence and probability. Dasgupta [19] investigated extensively the prime and primary hyperideals of multiplicative hyperrings with absorbing zero. Davvaz and Leoreanu-Fotea, in their monograph [20], presented the main results obtained in hyperring theory till the publication of it, but also an outline of applications of hyperstructures. Recently, Davvaz [21] published the first monograph on semihypergroup theory which covers most of the mathematical ideas and techniques required in the study of semihypergroups. Recently, in [22], r-hyperideals were mentioned as a generalization of r-ideals in commutative rings, and in [23], properties of r-hyperideals and some generalizations of them are investigated in the commutative Krasner hyperrings. Yeşilot et al. defined a hyperideal expansion and δ-primary hyperideal in [24]. Let $(\mathfrak{R}, \oplus,^\circ)$ be a commutative Krasner hyperring, with nonzero unit, δ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R})$ function is defined as a hyperideal expansion function that meets the following requirements where $L(\mathfrak{R})$ denotes all hyperideals of $\mathfrak{R}$: $i)$ $N \subseteq \delta(N)$, for all $N \in L(\mathfrak{R})$, $ii)$ If $N \subseteq M$, where N, $M \in L(\mathfrak{R})$, then $\delta(N) \subseteq \delta(M)$. Given an expansion δ of a hyperideal N of $\mathfrak{R}$ is called δ-primary if $a^\circ b \in N$, then $a \in N$ or $b \in \delta(N)$, for all $a, b \in \mathfrak{R}$. Ulucak [25] obtained some results on δ-primary and 2-absorbing δ-primary hyperideals.

In this paper, our goal is to extend the concept of δ-primary hyperideals to ϕ-δ-primary hyperideals in Krasner hyperrings and we intend to give some generalizations of hyperideals. Throughout this paper, $(\mathfrak{R}, \oplus,^\circ)$ will be a commutative Krasner hyperring with nonzero identity. We denote the set of all hyperideals of $\mathfrak{R}$ by $L(\mathfrak{R})$ and the set of all proper hyperideals of $\mathfrak{R}$ by $L^*(\mathfrak{R})$. We take ϕ as a function ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ at the sections of generalizations of prime and primary hyperideals. Firstly, we define ϕ-prime and ϕ-primary hyperideals in Krasner hyperrings and we give some characterizations for prime and primary hyperideals. Let ϕ be a function such that ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ and N be a hyperideal of $\mathfrak{R}$. N is said to be ϕ-prime (ϕ-primary) hyperideal of $\mathfrak{R}$ if $a^\circ b \in N - \phi(N)$, then $a \in N$ or $b \in N$ (resp., $b^n \in N$, for some $n \in \mathbb{N}$), for $a, b \in \mathfrak{R}$. Among other things, we give some characterizations in Theorem 2 and Theorem 5 as main theorems. Then we define ϕ-δ-primary hyperideals in Krasner hyperrings and also give several characterizations (See; Theorem 7 and Theorem 11). In this case, difference is that, ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ is a reduction function if $\phi(N) \subseteq N$ and $N \subseteq M$ implies $\phi(N) \subseteq \phi(M)$ for each $N, M \in L(\mathfrak{R})$. We investigate the attitude of ϕ-δ-primary hyperideal under homomorphism, in quotient ring, in Cartesian product and other cases (See; Theorem 10, Proposition 10, Proposition 9, Theorem 12).

2. Generalizations of Prime Hyperideals in Krasner Hyperrings

Throughout this section, $(\mathfrak{R}, \oplus,^\circ)$ is a commutative Krasner hyperring with nonzero identity. We denote the set of all hyperideals of $\mathfrak{R}$ by $L(\mathfrak{R})$. Initially, we give the definition of ϕ-prime hyperideal and some examples.

Definition 1. Let ϕ *be a function such that* ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ *and* N *be a hyperideal of* $\mathfrak{R}$. N *is said to be a* ϕ*-prime hyperideal of* $\mathfrak{R}$ *if* $a^\circ b \in N - \phi(N)$, *then* $a \in N$ *or* $b \in N$, *for* $a, b \in \mathfrak{R}$.

Example 1. Let $\mathfrak{R}$ *be a commutative Krasner hyperring. Consider the following functions* ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ *defined as follows*:

For any $N \in L(\mathfrak{R})$,

(i) $\phi_\varnothing(N) = \varnothing$.

(ii) $\phi_0(N) = 0$.

(iii) $\phi_2(N) = N^2$.

(iv) $\phi_n(N) = N^n$, *(for any* $n \geq 2$*)*

(v) $\phi_w(N) = \cap_{n=1}^{\infty} N^n$.

(vi) $\phi_1(N) = N$.

It is obvious that $\phi_\varnothing \leq \phi_0 \leq \phi_w \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \phi_{n-1} \leq \cdots \leq \phi_2 \leq \phi_1$.

Definition 2. Let $\mathfrak{R}$ *be a hyperring and* N *be a proper hyperideal of* $\mathfrak{R}$.

(i) N *is prime hyperideal if and only if it is* $\phi_\varnothing$*-prime hyperideal.*

(ii) N *is weakly prime hyperideal if and only if it is* ϕ_0*-prime hyperideal.*

(iii) *N is almost prime hyperideal if and only if it is ϕ_2-prime hyperideal.*

(iv) *N is n-almost prime hyperideal if and only if it is ϕ_n-prime hyperideal.*

(v) *N is w-prime hyperideal if and only if it is ϕ_w-prime hyperideal.*

Proposition 1. *Let N be a proper hyperideal of $\mathfrak{R}$.*

(1) If σ_1, σ_2 are two functions with $\sigma_1 \leq \sigma_2$ such that $\sigma_1, \sigma_2: L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ and N is σ_1-prime, then N is σ_2-prime.

(2) (i) N is prime hyperideal $\Rightarrow N$ is weakly prime hyperideal $\Rightarrow N$ is w-prime hyperideal $\Rightarrow N$ is $(n+1)$-almost prime hyperideal $\Rightarrow N$ is n-almost prime hyperideal for any $n \geq 2 \Rightarrow N$ is almost prime hyperideal.

(ii) N is w-prime hyperideal if and only if N is n-almost prime for any $n \geq 2$.

Proof

(1) Let us assume N is a σ_1-prime hyperideal of $\mathfrak{R}$. Take $a^{\circ}b \in N$ and $a^{\circ}b \notin \sigma_2(N)$, for $a, b \in \mathfrak{R}$. It follows $a^{\circ}b \notin \sigma_1(N)$ because of $\sigma_1 \leq \sigma_2$. Therefore $a^{\circ}b \in N - \sigma_1(N)$ and on the assumption that N is σ_1-prime hyperideal, then $a \in N$ or $b \in N$. That means N is σ_2-prime hyperideal of $\mathfrak{R}$.

(2) (*i*) We can obtain it by the ordering of the $\phi_i' s$ in Example 1 with the proof of 1.

(*ii*) Assume that N is w-prime. Let $a^{\circ}b \in N - \phi_w(N) = N - \cap_{n=1}^{\infty} N^n$. Then $a^{\circ}b \in N - \phi_n(N)$, since $\phi_w \leq \phi_n$, from (1) N is ϕ_n-prime. It means N is n-almost prime, for all $n \geq 2$. Conversely, suppose that N is n-almost prime, for $a^{\circ}b \in N - N^n$ for all $n \geq 2$. Thus $a^{\circ}b \in N - \cap_{n=2}^{\infty} N^n$. Hence $a^{\circ}b \in N - \cap_{n=1}^{\infty} N^n$. Therefore $a \in N$ or $b \in N$, since N is n-almost prime. □

Theorem 1. *Let $\phi: L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ be a function and T be a proper hyperideal of $\mathfrak{R}$ such that T is a ϕ-prime hyperideal of $\mathfrak{R}$. If T is not prime, then $T^2 \subseteq \phi(T)$. Hence at the same time it means if $T^2 \nsubseteq \phi(T)$, then T is prime.*

Proof. Assume that $T^2 \nsubseteq \phi(T)$. We need to prove that T is prime. Take $x^{\circ}y \in T$ for $x, y \in \mathfrak{R}$. If $x^{\circ}y \notin \phi(T)$, then $x^{\circ}y \in T - \phi(T)$. Since T is ϕ-prime, then $x \in T$ or $y \in T$. Let us suppose $x^{\circ}y \in \phi(T)$. We can presume $x^{\circ}T \nsubseteq \phi(K)$, so $x^{\circ}m \notin \phi(T)$ for some $m \in T$. Then $x^{\circ}(y \oplus m) \subseteq T - \phi(T)$. So $x \in T$ or $y \oplus m \subseteq T$, since T is ϕ-prime hyperideal and hence $x \in T$ or $y \in T$. Now on we can assume that $x^{\circ}T \subseteq \phi(T)$. (Similarly, $y^{\circ}T \subseteq \phi(T)$) We can find some $n, k \in T$ with $n^{\circ}k \notin \phi(T)$ because of $T^2 \nsubseteq \phi(T)$. Then $(x \oplus n) \circ (y \oplus k) \subseteq T - \phi(T)$. Since T is ϕ-prime hyperideal, then $(x \oplus n) \subseteq T$ or $(y \oplus k) \subseteq T$. So we have $x \in T$ or $y \in T$. Hence T is prime hyperideal of $\mathfrak{R}$. □

Corollary 1. *If T is a ϕ-prime hyperideal of $\mathfrak{R}$ with $\phi \leq \phi_3$, then T is w-prime hyperideal.*

Proof. We know that T is ϕ-prime hyperideal while T is prime, for every ϕ. Therefore T is w-prime hyperideal of $\mathfrak{R}$. Assume that T is not prime. By Theorem 1, $T^2 \subseteq (T) \subseteq T^3$. So $\phi(T) = T^n$ for every $n \geq 2$. Therefore T is n-almost prime hyperideal for every $n \geq 2$. Hence T is w-prime hyperideal of $\mathfrak{R}$.

Some characterizations of ϕ-prime hyperideals are provided. □

Theorem 2. *Let N be a proper hyperideal of the commutative Krasner hyperring $\mathfrak{R}$ and let $\phi: L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ be a function. Then the following statements hold:*

(i) N is ϕ-prime hyperideal of $\mathfrak{R}$.

(ii) For $a \in \mathfrak{R} - N$, $(N: a) = N \cup (\phi(N): a)$.

(iii) For $a \in \mathfrak{R} - N$, $(N: a) = N$ or $(N: a) = (\phi(N): a)$.

(iv) For each hyperideals K and L of $\mathfrak{R}$ such that $K^{\circ}L \subseteq N$ and $K^{\circ}L \nsubseteq \phi(N)$, we have $K \subseteq N$ or $L \subseteq N$.

Proof

(*i*)$\Rightarrow$(*ii*) Take $a \in \mathfrak{R} - N$. Suppose that $b \in (N: a)$, then $a^{\circ}b \in N$. If $a^{\circ}b \notin \phi(N)$, then $b \in N$ since N is ϕ-prime hyperideal of $\mathfrak{R}$. If $a^{\circ}b \in \phi(N)$, then $b \in (\phi(N): a)$. Hence $((N: a) \subseteq N \cup (\phi(N): a))$. Other side holds since the assumption of $\phi(N) \subseteq N$.

(*ii*)$\Rightarrow$(*iii*) It is obvious because of $(N: a)$ is a hyperideal of $\mathfrak{R}$.

(*iii*)$\Rightarrow$(*iv*) Let K and L be hyperideals of $\mathfrak{R}$ such that $K^{\circ}L \subseteq N - \phi(N)$. Assume that $K \nsubseteq N$. Then there exists an element $a \in K - N$, by (*iii*) we have $(N: a) = N$ or $(N: a) = (\phi(N): a)$. If $a^{\circ}L \nsubseteq \phi(N)$, then $L \nsubseteq (\phi(N): a)$. Since (*iii*) holds, then $L \subseteq (N: a) = N$. We are done. Assume that $a^{\circ}L \subseteq \phi(N)$. Since $K^{\circ}L \nsubseteq \phi(N)$, then we can choose an element $x \in K$ such that $x^{\circ}L \subseteq N - \phi(N)$. If $x \notin N$, then by (*iii*), $(N: x) = N$ or $(N: x) = (\phi(N): x)$. Since $L \subseteq (N: x)$, but $L \nsubseteq (\phi(N): x)$, then we conclude that $L \subseteq (N: x) = N$. Assume that $x \in N$. We have $a \oplus x \subseteq K - N$, and also note that $(a \oplus x) \circ L \subseteq N - \phi(N)$, since $a^{\circ}L \subseteq \phi(N)$ and $x^{\circ}L \nsubseteq \phi(N)$. Then $L \subseteq (N: a \oplus x) = N$ which completes the proof.

(*iv*)$\Rightarrow$(*i*) Let $a^{\circ}b \in N - \phi(N)$. Then $(a) \circ (b) \subseteq N$, but $(a) \circ (b) \nsubseteq \phi(N)$. So $(a) \subseteq N$ or $(b) \subseteq N$. Then $a \in N$ or $b \in N$.

In Theorem 1 we show that, if T is a ϕ-prime hyperideal and T is not prime, then $T^2 \subseteq \phi(T)$. In the following, a corollary of Theorem 2 is given, the proof of which is similar to Corollary 14 [1]. □

Corollary 2. *Let T be a ϕ-prime hyperideal that is not prime. Then $T^{\circ}\sqrt{\phi(T)} \subseteq \phi(T)$.*

Proof. Let $a \in \sqrt{\phi(T)}$. If $a \in T$, then $a^{\circ}T \subseteq T^2 \subseteq \phi(T)$ from Theorem 1. Let us suppose that $a \notin T$. From the main

Theorem 2, $(T: a) = T$ or $(T: a) = (\phi(T): a)$ as $T \subseteq (T: a)$ gives $a^{\circ}T \subseteq \phi(T)$. Let us suppose $(T: a) = T$. Assume that $a^n \in \phi(T)$, but $a^{n-1} \notin \phi(T)$. Then $a^n \in T$, so $a^{n-1} \in (T: a) = T$. Hence $a^{n-1} \in T - \phi(T)$, so $a \in T$, which is a contradiction.

In the following, we give a proposition about quotient and localization of a hyperring. Let S be a multiplicatively closed subset of a Krasner hyperring $\mathfrak{R}$. Define $\phi: L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ and $\phi_S: L(\mathfrak{R}_S) \longrightarrow L(\mathfrak{R}_S) \cup \{\varnothing\}$ with $\phi_S(M) = (\phi(M \cap \mathfrak{R}))_S$. Also $\phi_S(M) = \varnothing$, where. $\phi(M \cap \mathfrak{R}) = \varnothing$.

Let N, M be hyperideals of $\mathfrak{R}$ and $N \subseteq M$. Define $\phi_M: L(R/M) \longrightarrow L(R/M) \cup \{\varnothing\}$ with $\phi_M(N/M) = (\phi(N) \oplus M)/M$. Also $\phi_M(N/M) = \varnothing$, where $\phi(N) = \varnothing$. □

Proposition 2. *Suppose that* $\phi: L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ *is a function and* T *is a* ϕ*-prime hyperideal of* $\mathfrak{R}$.

(i) If M *is a hyperideal of* $\mathfrak{R}$ *with* $M \subseteq T$*, then* T/M *is* ϕ_M*-prime hyperideal of* $\mathfrak{R}/M$.

(ii) Let S *be a multiplicatively closed subset of* $\mathfrak{R}$ *with* $T \cap S = \varnothing$ *and* $\phi(T)_S \subseteq \phi_S(T_S)$*. Then* T_S *is a* ϕ_S*-prime hyperideal of* $\mathfrak{R}_S$.

Proof

(i) Let $x, y \in \mathfrak{R}$. Suppose that $(x \oplus M)\circ(y \oplus M) \subseteq (T/M) - \phi_M(T/M)$. So $x^{\circ} y \oplus M \subseteq T/M - (\phi(T \oplus M))/M \in$. Then $x^{\circ} y \in T - \phi(T \oplus M)$ We find that $x^{\circ} y \in T - \phi(T)$ and then $x \in T$ or $y \in T$. Therefore $x \oplus M \subseteq T/M$ or $y \oplus M \subseteq T/M$. So T/M is ϕ_M-prime hyperideal of $\mathfrak{R}/M$.

(ii) Let $(a/s)\circ(b/t) \in T_S - \phi_S(T_S)$, for some $a, b \in \mathfrak{R}$; $s, t \in S$. Then we have $p^{\circ} a^{\circ} b \in T$ for some $p \in S$ but $q^{\circ} a^{\circ} b \notin \phi(T) \cap \mathfrak{R}$ for every $q \in S$. Now if $q^{\circ} a^{\circ} b \in \phi(T)$, then $(a/s)\circ(b/t) \in \phi(T)_S \subseteq \phi_S(T_S)$ which is a contradiction. So $p^{\circ} a^{\circ} b \in T - \phi(T)$ and since T is ϕ-prime hyperideal of $\mathfrak{R}$, then we get either $p^{\circ} a \in T$ or $b \in T$. Hence $(a/s) \in T_S$ or $(b/t) \in T_S$, since $T \cap S = \varnothing$. Thus T_S is ϕ_S-prime hyperideal of $\mathfrak{R}_S$. □

Theorem 3

(i) Let X *and* Y *be commutative Krasner hyperrings and* N *be a weakly prime hyperideal of* X*. Then* $M = N \otimes Y$ θ *is a* ϕ*-prime hyperideal of* $\mathfrak{R} = X \otimes Y$*, for all* ϕ *with* $\phi_w \leq \phi \leq \phi_1$.

(ii) Suppose $\mathfrak{R}$ *is a commutative Krasner hyperring and* M *is a finitely generated proper hyperideal of* $\mathfrak{R}$*. Assuming* M *is a* ϕ*-prime hyperideal with* $\phi \leq \phi_3$*. Then* M *is either weakly prime or* $M^2 \neq 0$ *is idempotent and* $\mathfrak{R}$ *decomposes as* $X \otimes Y$ *where* $Y = M^2$ *and* $M = N \otimes Y$ *with* N *be weakly prime. As a result* M *is* ϕ*-prime for each* ϕ *with* $\phi_w \leq \phi \leq \phi_1$.

Proof

(i) Let N be a weakly prime hyperideal of X. Then $M = N \otimes Y$ not necessarily to be a weakly prime hyperideal of $\mathfrak{R} = X \otimes Y$; actually M is weakly prime hyperideal if and only if M is prime hyperideal. Nevertheless M is a ϕ-prime hyperideal for each ϕ with $\phi_w \leq \phi$. If N is prime, then M is prime and hence ϕ-prime for all ϕ. Assume that N is not prime. Then $N^2 = 0$. Therefore $M^2 = 0 \otimes Y$ and hereby $\phi_w(M) = 0 \otimes Y$. Hence we can write $M - \phi_w(M)$ in this form $M - \phi_w(M) = N \otimes Y - 0 \otimes Y = (N - \{0\}) \otimes Y$. Let we take $(a_1, a_2)\circ(b_1, b_2) = (a_1 \circ b_1, a_2 \circ b_2) \in M - \phi_w(M)$. It means $a_1 \circ b_1 \in N - \{0\}$, so $a_1 \in N$ or $b_1 \in N$. Then $(a_1, a_2) \in M$ or $(b_1, b_2) \in M$. Hence M is ϕ_w-prime, therefore ϕ-prime hyperideal.

(ii) If M is prime hyperideal, then M is weakly prime hyperideal of $\mathfrak{R}$. Suppose that M is not prime. From Theorem 1, $M^2 \subseteq \phi(M)$; and hence $M^2 \subseteq \phi(M) \subseteq \phi_3(M) = M^3$. So $M^2 = M^3$, it means M^2 is idempotent. Since M^2 is finitely generated, then $M^2 = (m)$ for some idempotent $m \in \mathfrak{R}$. Assume that $M^2 = 0$. Then $\phi(M) \subseteq M^3 = 0$. Hereby $\phi(M) = 0$. Consequently M is weakly prime hyperideal of $\mathfrak{R}$. Now suppose $M^2 \neq 0$. Take that $Y = M^2 = \mathfrak{R}^{\circ} m$ and $X = \mathfrak{R}^{\circ}(1 \ominus m)$, so $\mathfrak{R}$ decomposes as $X \otimes Y$ where $Y = M^2$. Let $I = M^{\circ}(1 \ominus m)$, so $M = N \otimes Y$ where $N^2 = (M^{\circ}(1 \ominus m))^2 = M^{2}\circ(1 \ominus m)^2 = (m)\circ(1 \ominus m) = 0$. To show N is weakly prime hyperideal, let $a^{\circ} b \in N^2 - \{0\}$; so $(a, 1) \circ(b, 1) = (a^{\circ} b, 1) \in N \otimes Y - (N \otimes Y)^2 = N \otimes Y - 0 \otimes Y \subseteq M - \phi(M)$. Since $\phi \leq \phi_3$, then it follows $\phi(M) \subseteq M^3 = (N \otimes Y)^3 = 0 \otimes Y$. We obtain that $(a, 1) \in M$ or $(b, 1) \in M$. Therefore $a \in N$ or $b \in N$. As a consequence N is weakly prime hyperideal of $\mathfrak{R}$. □

Proposition 3. *Let* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *be commutative Krasner hyperrings and let* $\varphi_i: L(\mathfrak{R}_i) \longrightarrow L(\mathfrak{R}_i) \cup \{\varnothing\}$ *be a function for* $i = 1, 2$*. Take* $\phi = \varphi_1 \times \varphi_2$ *and* $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$*. Then* N *is* ϕ*-prime hyperideal of* $\mathfrak{R}$ *if and only if* N *is one of the following types:*

(i) $N = N_1 \times N_2$*, where* N_i *is a proper hyperideal of* $\mathfrak{R}_i$ *with* $\varphi_i(N_i) = N_i$.

(ii) $N = N_1 \times \mathfrak{R}_2$*, where* N_1 *is* φ_1*-prime hyperideal of* $\mathfrak{R}_1$ *that should be prime if* $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$.

(iii) $N = \mathfrak{R}_1 \times N_2$*, where* N_2 *is* φ_2*-prime hyperideal of* $\mathfrak{R}_2$*- that should be prime if* $\varphi_1(\mathfrak{R}_1) \neq \mathfrak{R}_1$.

Proof. (⇒) (i) Obviously N is ϕ-prime hyperideal, since $N_1 \times N_2 - \phi(N_1 \times N_2) = \varnothing$

(ii) Let N_1 is φ_1-prime hyperideal of $\mathfrak{R}_1$ and $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$. Assume that $(a_1 \circ b_1, a_2 \circ b_2) = (a_1, a_2)\circ(b_1, b_2) \in N_1 \times \mathfrak{R}_2 - \varphi_1(N_1) \times \varphi_2(\mathfrak{R}_2) = (N_1 - \varphi_1(N_1)) \times (\mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2))$ for $(a_1, a_2), (b_1, b_2) \in \mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$. Thus $a_1 \circ b_1 \in N_1 - \varphi_1(N_1)$ and then either $a_1 \in N_1$ or $b_1 \in N_1$. So $(a_1, a_2) \in N_1 \times \mathfrak{R}_2$ or

$(b_1, b_2) \in N_1 \times \mathfrak{R}_2$. Therefore $N_1 \times \mathfrak{R}_2$ is ϕ-prime hyperideal of $\mathfrak{R}$.

(iii) The proof is similar to (ii).

$(\Leftarrow)$ Assume that N is a ϕ-prime hyperideal of $\mathfrak{R}$ where $\varphi_i(N_i) \neq N_i$. Let $a^\circ b \in N_1 - \varphi_1(N_1)$ for some $a, b \in \mathfrak{R}_1$. So $(a, 0)^\circ(b, 0) = (a^\circ b, 0) \in N - \phi(N)$. Since N is a ϕ-prime hyperideal of $\mathfrak{R}$, then $(a, 0) \in N$ or $(b, 0) \in N$. So $a \in N_1$ or $b \in N_1$. Therefore N_1 is φ_1-prime hyperideal of $\mathfrak{R}_1$. Similarly we can find N_2 is φ_2-prime hyperideal of $\mathfrak{R}_2$. Now we need to show that $N_1 = \mathfrak{R}_1$ or $N_2 = \mathfrak{R}_2$. Suppose that $N_2 \neq \mathfrak{R}_2$. Take $b_1 \in N_1 - \varphi_1(N_1)$, $b_2 \in \mathfrak{R}_2 - N_2$. Then note that $(1, 0)^\circ(b_1, b_2) = (b_1, 0) \in N - \phi(N)$. Since $N_2 \neq \mathfrak{R}_2$, then $(1, 0) \in N$ and so we find $1 \in N_1$. We get $N_1 = \mathfrak{R}_1$. Similarly one can easily find $N_2 = \mathfrak{R}_2$, if $N_1 \neq \mathfrak{R}_1$. Without loss of generality $N_1 \neq \mathfrak{R}_1$. Now let we show that N_1 is prime hyperideal with $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$. For $m' \in \mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2)$, let $x^\circ m \in N_1$ for some $x, m \in \mathfrak{R}_1$. Then we conclude that $(x, 1)^\circ(m, m\prime) = (x^\circ m, m\prime) \in N - \phi(N)$. Since N is a ϕ-prime hyperideal of $\mathfrak{R}$, then we find $(x, 1) \in N$ or $(m, m\prime) \in N$ which implies that $x \in N_1$ or $m \in N_1$. Therefore N_1 is a prime hyperideal of $\mathfrak{R}_1$. If $\varphi_1(\mathfrak{R}_1) \neq \mathfrak{R}_1$ and $N_1 = \mathfrak{R}_1$, then similarly one can prove that N_2 is a prime hyperideal of $\mathfrak{R}_2$. □

3. Generalizations of Primary Hyperideals in Krasner Hyperrings

Similar to the previous section, we consider $(\mathfrak{R}, \oplus, ^\circ)$ to be a commutative Krasner hyperring with nonzero unit. We denote the set of all hyperideals of $\mathfrak{R}$ by $L(\mathfrak{R})$. Let we define ϕ-primary hyperideal.

Definition 3. Let$\mathfrak{R}$be a commutative hyperring and$N$be a proper hyperideal of$\mathfrak{R}$.Letϕbe a function such thatϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$.$N$is called$\phi$-primary hyperideal of$\mathfrak{R}ifa^\circ b \in N - \phi(N)$, then either$a \in Norb^k \in N$for some$a, b \in \mathfrak{R}$, $k \in \mathbb{N}$.

Example 2. Let$\mathfrak{R}$be a commutative Krasner hyperring. Then we define functionsϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$such as in Example 1. *Also we have the same order,$\phi_\varnothing \leq \phi_0 \leq \phi_w \leq \ldots \leq \phi_{n+1} \leq \phi_n \leq \phi_{n-1} \leq \ldots \leq \phi_2 \leq \phi_1$.*

Definition 4. Let$\mathfrak{R}$be a hyperring and$N$be a proper hyperideal of$\mathfrak{R}$.

(i) *$N$is primary hyperideal if and only if it is$\phi_\varnothing$-primary hyperideal.*

(ii) *$N$is weakly primary hyperideal if and only if it isϕ_0-primary hyperideal.*

(iii) *$N$is almost primary hyperideal if and only if it isϕ_2-primary hyperideal.*

(iv) *$N$is$n$-almost primary hyperideal if and only if it isϕ_n-primary hyperideal.*

(v) *$N$is$w$-primary hyperideal if and only if it isϕ_w-primary hyperideal.*

Proposition 4. *Let$N$be a proper hyperideal of$\mathfrak{R}$.*

(1) Ifσ_1,$\sigma_2$are two functions with$\sigma_1 \leq \sigma_2$such thatσ_1, σ_2: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}andNis\sigma_1$-primary, then$Nis\sigma_2$-primary.

(2) (i)$N$is primary hyperideal$\Rightarrow N$is weakly primary hyperideal$\Rightarrow N$is$w$-primary hyperideal$\Rightarrow N$is$(n+1)$-almost primary hyperideal$\Rightarrow N$is$n$-almost primary hyperideal, for$n \geq 2 \Rightarrow N$is almost primary hyperideal.

(ii) N is w-primary hyperideal if and only if N is n-almost primary hyperideal for all $n \geq 2$.

Proof

(1) We suppose N is a σ_1-primary hyperideal of $\mathfrak{R}$. Take $a^\circ b \in N - \sigma_2(N)$ for $a, b \in \mathfrak{R}$. It means $a^\circ b \in N - \sigma_1(N)$. Because of N is σ_1-primary hyperideal of $\mathfrak{R}$, $a \in N$ or $b^k \in N$, for some $k \in \mathbb{N}$. Therefore N is σ_2-primary hyperideal.

(2) (i) We can obtain it by the ordering of the $\phi_i's$ given in 2. (ii) Assume that $a^\circ b \in N - N^n$ for all $n \geq 2$. Then $a^\circ b \in N - \cap_{n=2}^{\infty} N^n$ and we find $a^\circ b \in N - \cap_{n=1}^{\infty} N^n$. Since N is w-primary hyperideal of $\mathfrak{R}$, then we have $a \in N$ or $b^k \in N$, for some $k \in \mathbb{N}$. Conversely, if $a^\circ b \in N - \cap_{n=1}^{\infty} N^n$, then $a^\circ b \in N - N^n$ for some $n \geq 1$. Actually $a^\circ b \in N - N^n$ for some $n \geq 2$. Therefore $a \in N$ or $b^k \in N$ for some $k \in \mathbb{N}$, since N is n-almost primary for all $n \geq 2$.

The next theorem shows us how to determine ϕ-primary hyperideal to be primary. We call it as a characterization. □

Theorem 4. *Letϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$be a function and$T$be a proper hyperideal of$\mathfrak{R}$, such that$T$is aϕ-primary hyperideal of$\mathfrak{R}$. If$T$is not primary, then$T^2 \subseteq \phi(T)$. Hence at the same time it means if$T^2 \nsubseteq \phi(T)$, then$T$is primary hyperideal of$\mathfrak{R}$.*

Proof. Assume that $T^2 \nsubseteq \phi(T)$. We need to see that T is primary. Take some $x^\circ y \in T$ for $x, y \in \mathfrak{R}$. If $x^\circ y \notin \phi(T)$, then since T is ϕ-primary, $x \in T$ or $y^k \in T$ for some $k \in \mathbb{N}$. If $x^\circ y \in \phi(T)$, assuming that $x^\circ T \nsubseteq \phi(T)$, we get $x^\circ p_0 \notin \phi(T)$, where $p_0 \in T$. We have $x^\circ(y \oplus p_0) \subseteq T - \phi(T)$. Thus $x \in T$ or $(y \oplus p_0)^k \subseteq T$ for some $k \in \mathbb{N}$. It follows $x \in T$ or $y^k \in T$. Now we can assume that $x^\circ T \subseteq \phi(T)$. (In the same way we can assume that $y^\circ T \subseteq \phi(T)$). Because of $T^2 \nsubseteq \phi(T)$, there exist $p_1, q_1 \in T$ with $p_1{}^\circ q_1 \notin \phi(T)$. Then $(x \oplus p_1) \circ (y \oplus q_1) \subseteq T - \phi(T)$. As T is ϕ-primary, so $(x \oplus p_1) \subseteq T$ or $(y \oplus q_1)^m \subseteq T$ for some $m \in \mathbb{N}$. Consequently T is primary hyperideal of $\mathfrak{R}$. □

Corollary 3. *If$T$is aϕ-primary hyperideal of$\mathfrak{R}$, where$\phi \leq \phi_3$, then$T$is$w$-primary hyperideal.*

Proof. The proof is similar to Corollary 1.

In the following, some characterizations of ϕ-primary hyperideals are provided. □

Theorem 5. *Let N be a proper hyperideal of the commutative Krasner hyperring $\mathfrak{R}$ and let ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ be a function. The following statements hold:*

(i) *N is ϕ-primary hyperideal of $\mathfrak{R}$.*

(ii) *For $a \in \mathfrak{R} - \sqrt{N}$; $(N: a) = N \cup (\phi(N): a)$.*

(iii) *For $a \in \mathfrak{R} - \sqrt{N}$; $(N: a) = N$ or $(N: a) = (\phi(N): a)$.*

(iv) *For each hyperideals K and L of $\mathfrak{R}$, if $K \circ L \subseteq N$ and $K \circ L \nsubseteq \phi(N)$, then $K \subseteq N$ or $L \subseteq \sqrt{N}$*

Proof

$(i) \Rightarrow (ii)$ Let N is ϕ-primary hyperideal of $\mathfrak{R}$. It is obvious that $N \cup (\phi(N): a) \subseteq (N: a)$. To prove the other side; for every $b \in (N: a)$, we have $a \circ b \in N$. If $a \circ b \in N - \phi(N)$, then $b \in N$ since N is ϕ-primary and $a \in \mathfrak{R} - \sqrt{N}$. If $a \circ b \in \phi(N)$ then $b \in (\phi(N): a)$. So $(N: a) \subseteq N \cup (\phi(N): a)$. Hence $(N: a) = N \cup (\phi(N): a)$.

$(ii) \Rightarrow (iii)$ It is obvious because of $(N: a)$ is a hyperideal of $\mathfrak{R}$.

$(iii) \Rightarrow (iv)$ Let K and L be hyperideals of $\mathfrak{R}$, with $K \circ L \subseteq N - \phi(N)$. Assume that $L \subseteq \sqrt{N}$. Then there exists an element $a \in L - \sqrt{N}$ by (iii) we have $(N: a) = N$ or $(N: a) = (\phi(N): a)$. If $K \circ a \nsubseteq \phi(N)$, then $K \nsubseteq (\phi(N): a)$. Since (iii) holds, then $K \subseteq (N: a = N)$. We are done. Assume that $K \circ a \subseteq \phi(N)$. Since $K \circ L \subseteq \phi(N)$, then we can choose an element $x \in L$ such that $K \circ x \subseteq N - \phi(N)$. If $x \notin \sqrt{N}$, then by (iii), $(N: x) = N$ or $(N: x) = (\phi(N): x)$. Since $K \subseteq (N: x)$, but $K \subseteq (\phi(N): x)$, then we conclude that $K \subseteq (N: x) = N$. So assume that $x \in \sqrt{N}$. Then we have $a \oplus x \subseteq L - \sqrt{N}$, and also note that $K \circ (a \oplus x) \subseteq N - \phi(N)$, since $K \circ a \subseteq \phi(N)$ and $K \circ x \nsubseteq \phi(N)$. Then $K \subseteq (N: a \oplus x) = N$ which completes the proof.

$(iv) \Rightarrow (i)$ Let $a \circ b \in N - \phi(N)$. Then $(a) \circ (b) \subseteq N$, but $(a) \circ (b) \nsubseteq \phi(N)$. So $(a) \subseteq N$ or $(b) \subseteq \sqrt{N}$. Therefore $a \in N$ or $b^m \in N$ for some $m \in \mathbb{N}$.

In Theorem 4, we prove that if T is a ϕ-primary hyperideal that is not primary, then $T^2 \subseteq \phi(T)$. □

Proposition 5. *Let ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ be a function and T be a ϕ-primary hyperideal of $\mathfrak{R}$.*

(i) *If T is an hyperideal of $\mathfrak{R}$ with $M \subseteq T$, then T/M is ϕ_M-primary hyperideal of $\mathfrak{R}/M$.*

(ii) *Let S be a multiplicatively closed subset of $\mathfrak{R}$ with $T \cap S = \varnothing$ and $\phi(T)_S \subseteq \phi(T_S)$. Then T_S is an ϕ_S-primary hyperideal of $\mathfrak{R}_S$.*

Proof

(i) Let $x, y \in \mathfrak{R}$. We assume that $(x \oplus M) \circ (y \oplus M) \subseteq T/M - \phi_M(T/M)$, it means $x \circ y \oplus M \subseteq T/M - (\phi(T \oplus M))/M$. We find $x \circ y \in T - \phi(T \oplus M)$. Then $x \circ y \in T - \phi(T)$ and so $x \in T$ or $y^k \in T$ for some $k \in \mathbb{N}$. Hence $x \oplus M \subseteq T/M$ or $(y \oplus M)^k \subseteq T/M$. This gives us T/M is ϕ_M-primary hyperideal of $\mathfrak{R}/M$.

(ii) Let $(a/s) \circ (b/t) \in T_S - \phi_S(T_S)$, for some $a, b \in \mathfrak{R}$; $s, t \in S$. Then we have $p \circ a \circ b \in T$ for some $p \in S$ but $q \circ a \circ b \notin \phi(T) \cap \mathfrak{R}$ for every $q \in S$. Now if $q \circ a \circ b \in \phi(T)$, then $(a/s) \circ (b/t) \in \phi(T)_S \subseteq \phi_S(T_S)$ gives us a contradiction. So $p \circ a \circ b \in T - \phi(T)$ and since T is ϕ-primary, then we get either $p \circ a \in T$ or $b^k \in T$ for some $k \in \mathbb{N}$. Therefore $a/s \in T_S$ or $b^k/t^k \in T_S$ because of $\sqrt{T} \cap S = \varnothing$ it is same with $T \cap S = \varnothing$). Thus T_S is ϕ_S-primary hyperideal of $\mathfrak{R}_S$. □

Theorem 6

(i) *Let X, Y be commutative Krasner hyperrings and N be a weakly primary hyperideal of X. Then $M = N \otimes Y$ is a ϕ-primary hyperideal of $\mathfrak{R} = X \otimes Y$, for all ϕ with $\phi_w \leq \phi \leq \phi_1$.*

(ii) *Let $\mathfrak{R}$ be a commutative Krasner hyperring and M be a finitely generated proper hyperideal of $\mathfrak{R}$, such that M is a ϕ-primary hyperideal with $\phi \leq \phi_3$. Then M is either weakly primary or $M^2 \neq 0$ is idempotent and $\mathfrak{R}$ decomposes as $X \otimes Y$, where $Y = M^2$ and $M = N \otimes Y$, where N is weakly primary. As a result M is ϕ-primary for each ϕ with $\phi_w \leq \phi \leq \phi_1$.*

Proof

(i) Let N be a weakly primary hyperideal of X. M is weakly primary if and only if M is primary. Nevertheless M is a ϕ-primary hyperideal for each ϕ with $\phi_w \leq \phi$. If N is primary, then M is primary and hence ϕ-primary for all ϕ. Assume that N is not primary. Then $N^2 = 0$. Therefore $M^2 = 0 \otimes Y$ and hereby $\phi_w(M) = 0 \otimes Y$. Hence we can write $M - \phi_w(M)$ in this form $M - \phi_w(M) = N \otimes Y - 0 \otimes Y = (N - \{0\}) \otimes Y$. Let we take $(a_1, a_2) \circ (b_1, b_2) = (a_1 \circ b_1, a_2 \circ b_2) \in M - \phi_w(M)$. It means $a_1 \circ b_1 \in N - \{0\}$, so $a_1 \in N$ or $b_1^s \in N$ for some $s \in \mathbb{N}$. Then $(a_1, a_2) \in N \otimes Y$ or $(b_1, b_2)^s \in N \otimes Y$. Hence $M = N \otimes Y$ is ϕ_w-primary, therefore M is ϕ -primary hyperideal.

(ii) If M is primary hyperideal, then M is weakly primary hyperideal of $\mathfrak{R}$. Suppose that M is not primary. From Theorem 4, $M^2 \subseteq \phi(M)$ and hence $M^2 \subseteq \phi(M) \phi_3(M) = M^3$. So $M^2 = M^3$, which means M^2 is idempotent. Since M^2 is finitely generated, then $M^2 = (m)$ for some idempotent $m \in \mathfrak{R}$. Assume that $M^2 = 0$. Then $\phi(M) \subseteq M^3 = 0$. Hereby $\phi(M) = 0$. Consequently, M is weakly primary hyperideal of $\mathfrak{R}$. Now let us suppose $M^2 \neq 0$. Take $Y = M^2 = \mathfrak{R} \circ m$ and $X = \mathfrak{R} \circ (1 \ominus m)$, so $\mathfrak{R}$ decomposes as $X \otimes Y$, where $Y = M^2$. Let $X = M \circ (1 \ominus m)$, so $M = N \otimes Y$, where $N^2 = (M \circ (1 \ominus m))^2 = M^2 \circ (1 \ominus m)^2 = (m)$

$\circ (1 \ominus m) = 0$. To show N is weakly primary hyperideal, let $a \circ b \in N^2 - \{0\}$. Thus $(a, 1) \circ (b, 1) = (a \circ b, 1) \in N \otimes Y - (N \otimes Y)^2 = N \otimes Y - 0 \otimes Y \subseteq M - \phi(M)$, since $\phi \leq \phi_3$ implies $\phi(M) \subseteq M^3 = (N \otimes Y)^3 = 0 \otimes Y$. We obtain that $(a, 1) \in M$ or $(b, 1)^t \in M$, for some $t \in \mathbb{N}$. Therefore $a \in N$ or $b^t \in N$. As a consequence, N is weakly primary hyperideal of $\mathfrak{R}$. □

Proposition 6. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be commutative Krasner hyperrings and let φ_i: $L(\mathfrak{R}_i) \longrightarrow L(\mathfrak{R}_i) \cup \{\varnothing\}$ be a function, for $i = 1, 2$. Take $\varphi = \varphi_1 \times \varphi_2$ and $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$. Then N is ϕ-primary hyperideal of $\mathfrak{R}$ if and only if N is one of the following types:*

(i) $N = N_1 \times N_2$ where N_i is a proper hyperideal of $\mathfrak{R}_i$ with $\varphi_i(N_i) = N_i$.

(ii) $N = N_1 \times \mathfrak{R}_2$ where N_1 is φ_1-primary hyperideal of $\mathfrak{R}_1$ that should be primary if $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$.

(iii) $N = \mathfrak{R}_1 \times N_2$ where N_2 is φ_2-primary hyperideal of $\mathfrak{R}_2$ that should be primary if $\varphi_1(\mathfrak{R}_1) \neq \mathfrak{R}_1$.

Proof. (⇒) (i) Obviously N is ϕ-primary hyperideal, since $N_1 \times N_2 - \phi(N_1 \times N_2) = \varnothing$.

(ii) Let N_1 be φ_1-primary hyperideal of $\mathfrak{R}_1$ and $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$. Assume that $(a_1 \circ b_1, \ a_2 \circ b_2) = (a_1, a_2) \circ (\ b_1, b_2) \in \ N_1 \times \mathfrak{R}_2 - \varphi_1(N_1) \times \varphi_2(\mathfrak{R}_2) = (N_1 - \ \varphi_1(N_1)) \times (\mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2))$, for $(a_1, a_2), (b_1, b_2) \in \mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$. We have $a_1 \circ b_1 \in N_1 - \varphi_1(N_1)$, then either $a_1 \in N_1$ or $b_1^k \in N_1$, for some $k \in \mathbb{N}$, since N_1 is φ_1-primary hyperideal of $\mathfrak{R}_1$. So $(a_1, a_2) \in N_1 \times \mathfrak{R}_2$ or $(b_1^k, b_2^k) = (b_1, b_2) \in N_1 \times \mathfrak{R}_2$. Therefore $N_1 \times \mathfrak{R}_2$ is ϕ-primary hyperideal of $\mathfrak{R}$.

(iii) The proof is similar to (ii).

(⇐) Suppose that N is a ϕ-primary hyperideal of $\mathfrak{R}$, where $\varphi_i(N_i) \neq N_i$. Let $a \circ b \in N_1 - \varphi_1(N_1)$, for some $a, b \in \mathfrak{R}_1$. Thus $(a, 0) \circ (b, 0) = (a \circ b, 0) \in N - \phi(N)$. Since N is a ϕ-primary hyperideal of $\mathfrak{R}$, then $(a, 0) \in N$ or $(b, 0)^k \in N$ for some $K \in \mathbb{N}$. So $a \in N_1$ or $b^k \in N_1$. Therefore N_1 is φ_1 -primary hyperideal of $\mathfrak{R}_1$. Similarly we can find N_2 is φ_2-primary hyperideal of $\mathfrak{R}_2$. We have to show now that $N_1 = \mathfrak{R}_1$ or $N_2 = \mathfrak{R}_2$. Suppose that $N_2 \neq \mathfrak{R}_2$. Let take $b_1 \in N_1 - \varphi_1(N_1)$, $b_2 \in \mathfrak{R}_2 - N_2$. Then note that $(1, 0) \circ (b_1, b_2) = (b_1, 0) \in N - \phi(N)$. Since $N_2 \neq \mathfrak{R}_2$, $(1, 0)^t = (1, 0) \in N$ $(N_2 \neq \mathfrak{R}_2)$, then we find $1 \in N_1$ and so we get $N_1 = \mathfrak{R}_1$. Similarly one can easily find $N_2 = \mathfrak{R}_2$, if $N_1 \neq \mathfrak{R}_1$. Without loss of generality $N_1 \neq \mathfrak{R}_1$. Now let we show that N_1 is primary hyperideal with $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$. For $m' \in \mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2)$, let $x \circ m \in N_1$, for some $x, m \in \mathfrak{R}_1$. Then we conclude that $(x, 1) \circ (m, m') = (x \circ m, m') \in N - \phi(N)$. Since N is a ϕ-primary hyperideal of $\mathfrak{R}$, then we find $(x, 1)^S \in N$ for some $s \in \mathbb{N}$ or $(m, m') \in N$ which implies that $x^s \in N_1$ or $m \in N_1$. Therefore N_1 is a primary hyperideal of $\mathfrak{R}_1$.

If $\varphi_1(\mathfrak{R}_1) \neq \mathfrak{R}_1$ and $N_1 = \mathfrak{R}_1$, then similarly one can prove that N_2 is a primary hyperideal of $\mathfrak{R}_2$. □

4. ϕ-δ-Primary Hyperideals in Krasner Hyperrings

Let N be a proper hyperideal of hyperring $\mathfrak{R}$. Denote the set of all hyperideals of $\mathfrak{R}$, by $L(\mathfrak{R})$ and denote the set of all proper hyperideals of $\mathfrak{R}$, by $L^*(\mathfrak{R})$. The function ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ is said to be reduction function if $\phi(N) \subseteq N$ and $N \subseteq M$ implies $\phi(N) \subseteq \phi(M)$, for each $N, M \in L(\mathfrak{R})$ and δ be an expansion function such that δ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R})$. Now, we give some examples related to reduction and expansion functions.

Example 3. Let $\mathfrak{R}$ be a commutative Krasner hyperring with a nonzero identity. Let us consider the following functions δ on $L(\mathfrak{R})$, for any $N \in L(\mathfrak{R})$,

(i) $\delta_0(N) = N$, *i.e., δ is the identity function.*

(ii) $\delta_1(N) = \sqrt{N}$, *i.e., δ is the radical operation.*

(iii) $\delta_{res}(N) = (N\colon M)$ *for a fixed* $M \in L(\mathfrak{R})$.

(iv) $\delta_{ann}(N) = ann(ann(N))$.

(vi) $\delta_M(N) = N \oplus M$ *for a fixed* $M \in L(\mathfrak{R})$.

All the above functions are examples of expansion on $L(\mathfrak{R})$.

Example 4. Let $\mathfrak{R}$ be a commutative Krasner hyperring with a nonzero identity. Consider the following functions ϕ: $L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ defined as follows: for any $N \in L(\mathfrak{R})$:

(i) $\phi_\varnothing(N) = \varnothing$.

(ii) $\phi_\varnothing(N) = 0$.

(iii) $\phi_1(N) = N$.

(iv) $\phi_2(N) = N^2$.

(v) $\phi_k(N) = N^k$.

(vi) $\phi_k(N) = \cap_{i=1}^{\infty} N^i$.

All the above functions are reduction on $L(\mathfrak{R})$. Remember that $\phi_\varnothing \leq \phi_0 \leq \phi_w \leq \ldots \leq \phi_{n+1} \leq \phi_n \leq \phi_{n+1} \leq \ldots \leq \phi_2 \leq \phi_1$.

Definition 5. Let δ be a hyperideal expansion, ϕ be a hyperideal reduction and N be a proper hyperideal of $\mathfrak{R}$. N is said to be a ϕ-δ-primary hyperideal if $a \circ b \in N - \phi(N)$, then either $a \in N$ or $b \in \delta(N)$ for each $a, b \in \mathfrak{R}$.

Remark 1. [24] *If $\delta_1, \delta_2, \ldots, \delta_n$ are hyperideal expansions, then $\delta = \cap_{i=1}^{n} \delta_i$ is also an hyperideal expansion.*

Definition 6. Let $\mathfrak{R}$ be a hyperring and N be a proper hyperideal of $\mathfrak{R}$

(i) *N is prime hyperideal if and only if it is $\phi_\varnothing$-δ_0-primary hyperideal* [24].

(ii) *N is primary hyperideal if and only if it is $\phi_{\varnothing}$-δ_1-primary hyperideal* [24].

(iii) *N is ϕ-prime hyperideal if and only if it is φ-δ_0-primary hyperideal.*

(iv) *N is ϕ-primary hyperideal if and only if it is φ-δ_1-primary.*

(v) *N is δ-primary hyperideal if and only if it is $\varphi_{\varnothing}$-δ-primary hyperideal.*

(vi) *N is weakly prime hyperideal if and only if it is ϕ_0-δ_0-primary hyperideal.*

(vii) *N is almost prime hyperideal if and only if it is φ_2-δ_0-primary hyperideal.*

Suppose that ϕ, σ are reductions on $L(\mathfrak{R})$. Then we write $\varphi \leq \sigma$ if $\phi(N) \subseteq \sigma(N)$, for all $N \in L(\mathfrak{R})$. Similarly, for any two expansions δ, γ on $L(\mathfrak{R})$; $\delta \leq \gamma$ if $\delta(N) \subseteq \gamma(N)$, for each $N \in L(\mathfrak{R})$.

Definition 7. Let $\mathfrak{R}$ be a hyperring and N be a proper hyperideal of $\mathfrak{R}$.

(i) *If N is ϕ_0-δ-primary hyperideal, then N is said to be a weakly δ-primary hyperideal of $\mathfrak{R}$.*

(ii) *If N is ϕ_2-δ-primary hyperideal, then N is said to be an almost δ-primary hyperideal of $\mathfrak{R}$.*

(iii) *If N is ϕ_n-δ-primary hyperideal, then N is said to be an r_1-almost δ-primary hyperideal of $\mathfrak{R}$.*

(iv) *If N is ϕ_w-δ-primary hyperideal, then N is said to be a w-δ-primary hyperideal of $\mathfrak{R}$.*

Proposition 7. *Let $\mathfrak{R}$ be a hyperring, N be a proper hyperideal of $\mathfrak{R}$, ϕ, φ be reductions on $L(\mathfrak{R})$ and δ, γ be expansions on $L(\mathfrak{R})$. The following statements hold:*

(i) If $\phi \leq \sigma$, then every ϕ-δ-primary hyperideal is a σ-δ-primary hyperideal.

(ii) If $\delta \leq \gamma$, then every ϕ-δ-primary hyperideal is a ϕ-γ-primary hyperideal.

(iii) Every ϕ-prime hyperideal is a ϕ-δ-primary hyperideal.

(iv) Every δ-primary hyperideal is a ϕ-δ-primary hyperideal.

(v) Suppose that N is proper hyperideal of $\mathfrak{R}$. N is weakly δ-primary $\Rightarrow$ N is w-δ-primary hyperideal $\Rightarrow$ N is r_1-almost δ-primary, for each $n \geq 2$ $\Rightarrow$ N is almost δ-primary.

Proof. $(i)(ii)$: Straightforward.

(iii): It follows from (ii) and Definition 6 (iii), since $\delta_0 \leq \delta$.

(iv): It follows from (i) and Definition 6 (iii), since $\phi_{\varnothing} \leq \phi$.

(v): It follows from (i) and the fact that $\phi_{\varnothing} \leq \phi_w \leq \phi_n \leq \phi_2$. □

Proposition 8. *Let ϕ be a hyperideal reduction, δ be a hyperideal expansion and $\{m_i\colon i \in \Delta\}$ be a directed family of ϕ-δ-primary hyperideals of $\mathfrak{R}$. Then $M = \cup_{ie\Delta} M_i$ is a ϕ-δ-primary hyperideal.*

Proof. Let $\{m_i\colon i \in \Delta\}$ be a directed family of ϕ-δ -primary hyperideals of $\mathfrak{R}$. Assume that $a \circ m \in M - \phi(\cup_{ie\Delta} M_i)$. This implies that $a \circ m \in M_i - \phi(M_i)$, for some $i \in \Delta$. We get either $a \circ m \in M_i$ or $m \in \delta(M_i)$, because of M_1 is a ϕ-δ-primary. If $a \in M_i$, then clearly we have $a \in \cup_{ie\Delta} M_i$. If $m \in \delta(M_i)$, then we have $m \in \delta(\cup_{ie\Delta} M_i)$, since $M \subseteq \cup_{ie\Delta} M_i$. Hence $M = \cup_{ie\Delta} M_i$ is a ϕ-δ-primary hyperideal.

In the following, we give a characterization for ϕ-δ-primary hyperideals such that ϕ is a hyperideal reduction and δ is a hyperideal expansion. □

Theorem 7. *Let $\mathfrak{R}$ be a Krasner hyperring and N be a proper hyperideal of $\mathfrak{R}$. Then the following statements hold:*

(i) N is a ϕ-δ-primary hyperideal;

(ii) For each $a \in \mathfrak{R} - \delta(N)$, $(N\colon a) = N \cup (\phi(N)\colon a)$;

(iii) For each $a \in \mathfrak{R} - \delta(N)$, $(N\colon a) = N$ or $(N\colon a) = (\phi(N)\colon a)$;

(iv) For each hyperideal K, L of $\mathfrak{R}$, $K \circ L \subseteq N$ and $K \circ L \subseteq \phi(N)$ imply that $K \subseteq N$ or $L \subseteq \delta(N)$.

(v) For each hyperideal M of $\mathfrak{R}$ such that $M \nsubseteq \delta(N)$, then $(N\colon M) = N$ or $(N\colon M) = (\phi(N)\colon M)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a ϕ-δ-primary hyperideal and $a \in \mathfrak{R} - \delta(N)$. It is clear that $N \cup (\phi(N)\colon a) \subseteq (N\colon a)$. Let $m \in (N\colon a)$. Then we have $a \circ m \in N$. If $a \circ m \in \phi(N)$, then we obtain $m \in (\phi(N)\colon a) \subseteq N \cup (\phi(N)\colon a)$. Assume that $a \circ m \notin \phi(N)$. Since $a \circ m N - \phi(N)$ and $a \notin \delta(N)$, then we get $m \in N \subseteq N \cup (\phi(N)\colon a)$. Hence, $(N\colon a) = N \cup (\phi(N)\colon a)$.

$(ii) \Rightarrow (iii)$ It follows from the fact that a hyperideal is a union of two hyperideals. Then it must be equal to one of them.

$(iii) \Rightarrow (iv)$: Let $K \circ L \subseteq N$, for some hyperideals K and L of $\mathfrak{R}$. Assume that $L \nsubseteq \delta(N)$ and $K \nsubseteq N$. Then there exists $m \in L - \delta(N)$. We have to show $K \circ L \subseteq \phi(N)$. By (iii), we have either $(N\colon m) = N$ or $(N\colon m) = (\phi(N)\colon m)$. If $K \circ m \subseteq N$, then by (iii), we get $K \subseteq (N\colon m) = N$ For $a \in K - N$, it means $a \in (N\colon m) - N$. Hence by part (iii), $(N\colon m) = (\phi(N)\colon m)$. Thus $K \subseteq (N\colon m) = (\phi(N)\colon m)$ implies that $K \circ m \subseteq \phi(N)$. On the other side, suppose that $m \in \delta(N)$. Then $m \in L \cup \delta(N)$. Choose an element $m' \in L - \delta(N)$ so $m \oplus m' \subseteq L - \delta(N)$. Hence $K \circ m' \subseteq \phi(N)$ and $K \circ (m \oplus m') \subseteq \phi(N)$. Let $a \in K$. Then $a(m \oplus m') \ominus a \circ m' = a \circ m' \subseteq \phi(N)$. Thus $K \circ m \subseteq \phi(N)$. Therefore $K \circ L \subseteq \phi(N)$.

$(iv) \Rightarrow (v)$: Suppose that M is a hyperideal of $\mathfrak{R}$ such that $M \nsubseteq \delta(N)$. Also, note that $M \circ (N\colon M) \subseteq N$. If $M \circ (N\colon M) \subseteq \phi(N)$, then we have $(N\colon M) \subseteq (\phi(N)$

$: M)\subseteq(N: M)$. Assume that $M\circ(N: M)\nsubseteq\phi(N)$. Then by (iv), $(N: M)\subseteq N\subseteq(N: M)$.

$(v)\Rightarrow(i)$: Suppose that $a\circ m\in N-\phi(N)$, with $a\notin\delta(N)$, for $a,m\in\mathfrak{R}$. Put $\mathfrak{R}\circ a=M$ and note that $m\in(N: M)$. Then by (v), we get either $m\in(N: M)=N$ or $m\in(N: M)=(\phi(N): M)$. The latter case is impossible, since $a\circ m\notin\phi(N)$. Therefore, N is a ϕ-δ-primary hyperideal. □

Theorem 8

(i) Let T be a ϕ-δ-primary hyperideal of $\mathfrak{R}$ such that $(\phi(T): a)=\phi(T: a)$, for each $a\in\mathfrak{R}$. Then $(T: a)$ is a ϕ-δ-primary hyperideal of $\mathfrak{R}$.

(ii) Suppose that $\delta\leq\delta_1$ are hyperideal expansions and T is a ϕ-δ-primary hyperideal of $\mathfrak{R}$ such that $\delta_1(\phi(T))\subseteq\delta(T)$. Then $\delta(T)=\delta_1(T)$.

Proof

(i) Let T be a ϕ-δ-primary hyperideal of $\mathfrak{R}$ such that $(\phi(T): a)=\phi(T: a)$, for each $a\in\mathfrak{R}$. We show that $(T: a)$ is a ϕ- δ-primary hyperideal of $\mathfrak{R}$. For this, let we take $x,m\in\mathfrak{R}$ such that $x\circ m\in(T: a)-\phi(T: a)$. Then we have $x\circ a\circ m\in T$. Since $(\phi(T): a)=\phi(T: a)$, then we also have $x\circ a\circ m\notin\phi(T)$. Since T is a ϕ-δ-primary hyperideal, then we get either $x\in\delta(T)$ or $a\circ m\in T$ implying either $x\in\delta(T: a)$ or $m\in(T: a)$. Therefore, $(T: a)$ is a ϕ-δ -primary hyperideal of $\mathfrak{R}$.

(ii) Since $\delta\leq\delta_1$ then we have $\delta(T)\subseteq\sqrt{T}=\delta_1(T)$. For the converse, take $x\in\sqrt{T}$. Then there exists minimal $k\in\mathbb{N}$ such that $x^k\in T$, that is, $x^{k-1}\notin T$. If $K=1$, then we have $x\in T\subseteq\delta(T)$. Now, assume that $K>1$. We have two cases. Case 1: Let $x\in\delta_1(\phi(T))=\sqrt{\phi(T)}$. Then we have $x\in\delta(T)$. Case 2: Let $x\notin\sqrt{\phi(T)}$. Then we have $x^k\notin\sqrt{\phi(T)}$, that is, $x^{k-1}\circ(x\circ\mathfrak{R})\subseteq T$ and $x\circ(x^{k-1}\circ\mathfrak{R})\nsubseteq\phi(T)$. Since T is a ϕ-δ -primary hyperideal of $\mathfrak{R}$, then by Theorem 7, $x\in\delta(T)$ or $x^{K-1}\circ\mathfrak{R}\subseteq T$. The latter case is impossible, since $x^{K-1}\notin T$. Thus in both cases, we have $x\in\delta(T)$. Therefore, $\delta(T)=\delta_1(T)$. □

Theorem 9. *Let T be a ϕ-δ-primary hyperideal of $\mathfrak{R}$ such that $\delta(T)\circ T\subseteq\phi(T)$. Then T is a δ-primary hyperideal of $\mathfrak{R}$.*

Proof. Let $a\circ m\in T$, for some $a,m\in\mathfrak{R}$. If $a\circ m\notin\phi(T)$, then we conclude either $a\in\delta(T)$ or $m\in T$ as T is an ϕ-δ -primary hyperideal of $\mathfrak{R}$. Assume that $a\circ m\in\phi(T)$. If $a\circ T\nsubseteq\phi(T)$, then there exists $n\in T$ such that $a\circ n\notin\phi(T)$. Thus we have $a\circ(m\oplus n)\subseteq T-\phi(T)$, which implies either $a\in\delta(T)$ or $m\oplus n\subseteq T$. Then we get $a\in\delta(T)$ or $m\in T$, which completes the proof. Assume that $a\circ T\subseteq\phi(T)$. Similarly, we may assume that $\delta(T)\circ m\subseteq\phi(T)$. As $\delta(T)\circ T\subseteq\phi(T)$, we can find $b\in\delta(T)$ and $m'\in T$ such that $b\circ m'\subseteq\phi(T)$. Then we conclude that $(a\oplus b)\circ(m\oplus m)'\subseteq T-\phi(T)$. Since T is a ϕ-δ-primary hyperideal of $\mathfrak{R}$, then we have either $a\oplus b\subseteq\delta(T)$ or $m\oplus m\subseteq T$, which implies that $a\in\delta(T)$ or $m\in T$. Therefore, T is a δ-primary hyperideal of $\mathfrak{R}$. □

Definition 8

(i) [24] *A hyperideal expansion δ is said to be global if for any hyperring good homomorphism μ: $\mathfrak{R}\longrightarrow\widehat{S}$, $\delta(\mu^{-1}(M))=\mu^{-1}(\delta(M))$, for each $M\in L(\widehat{S})$.*

(ii) *A hyperideal reduction ϕ is said to be a global if for any homo-*
morphism μ: $\mathfrak{R}\longrightarrow\widehat{S}$, $\phi(\mu^{-1}(M))=\mu^{-1}(\delta(M))$, for each $M\in L(\widehat{S})$.

For instance, the hyperideal reductions ϕ_0,ϕ_1 and the hyperideal expansions δ_0,δ_1 are both *global.*

Theorem 10. *Let μ be a good homomorphism from Krasner hyperring $(\mathfrak{R},\oplus,\circ)$ into a Krasner hyperring $(\widehat{S},+,\cdot)$. The following statements hold:*

(i) Let μ: $\mathfrak{R}\longrightarrow\widehat{S}$ be a good homomorphism and M be a ϕ-δ-primary hyperideal of $\widehat{S}$ such that ϕ is global reduction function and δ is global expansion function. Then $\mu^{-1}(M)=\mathfrak{R}$ or $\mu^{-1}(M)$ is a ϕ-δ-primary hyperideal of $\mathfrak{R}$.

(ii) Let μ: $\mathfrak{R}\longrightarrow\widehat{S}$ be a good epimorphism and N be a hyperideal of $\mathfrak{R}$ containing $Ker(\mu)$. Suppose that ϕ is global reduction function and δ is global expansion function. Then N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$ if and only if $\mu(N)$ is a ϕ-δ-primary hyperideal of $\widehat{S}$.

Proof

(i) Let μ: $\mathfrak{R}\longrightarrow\widehat{S}$ be a hyperring homomorphism and M be a ϕ-δ-primary hyperideal of $\widehat{S}$ such that $\mu^{-1}(M)\neq\mathfrak{R}$. Assume $a\circ m\in\mu^{-1}(M)-\phi(\mu^{-1}(M))$, for some $a,m\in\mathfrak{R}$. Since ϕ is global, then $\phi(\mu^{-1}(M))=\mu^{-1}(\phi(M))$ and thus $a\circ m\in\mu^{-1}(M)-(\mu^{-1}\phi(M))$. This implies that $\mu(a\circ m)=\mu(a)\cdot\mu(m)\in M=\phi(M)$. Since M is a ϕ-δ-primary hyperideal of $\widehat{S}$, then we have either $\mu(a)\in\delta(M)$ or $\mu(m)\in M$. Since $M\subseteq\mu^{-1}(M)$, then we get $a\in\mu^{-1}(\delta(M))=\delta(\mu^{-1}(M))$ or $M\subseteq\mu^{-1}(M)$. Therefore, $\mu^{-1}(M)$ is a ϕ-δ-primary hyperideal.

(ii) Let us suppose that $\mu(N)$ is a ϕ-δ-primary hyperideal of $\widehat{S}$, where N is a hyperideal of $\mathfrak{R}$ containing $\mathrm{Ker}(\mu)$. By (i), $\mu^{-1}(\mu(N))=N$ is a ϕ-δ-primary hyperideal of $\mathfrak{R}$. For the converse, let N be a ϕ-δ-primary hyperideal of $\mathfrak{R}$ and $a'\cdot m'\in\mu(N)-\phi(\mu(N))$, for some $a',m'\in\widehat{S}$. Then $\exists a,m\in\mathfrak{R}$ with $\mu(a)=a'$ and $\mu(m)=m'$, since μ is surjective. $a'\cdot m'=\mu(a)\cdot\mu(m)=\mu(a^\circ m)=\mu(x)\in\mu(N)-\phi(\mu(N))$, for some $x\in\mathfrak{R}$. So $0\in\mu(a^\circ m)\ -\mu(x)\ =\mu(a^\circ m\ominus x)$. Hence there exists $t\in a^\circ m\ominus x$ such that $\mu(t)=0_{\widehat{S}}$. $a^\circ m\in t\oplus x\subseteq\mathrm{Ker}(\mu)+N\subseteq N+N\subseteq N$. Then ϕ is global and $\mathrm{Ker}(\mu)\subseteq N$, we have $\phi(\mu(N))=\mu(\phi(N))$ and also $a^\circ m\in N-\phi(N)$.

Since N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$, then we have either $a \in \delta(N)$ or $m \in N$. Since $N \subseteq \mu(N)$, then we have $a' = \mu(a) \in \delta(\mu(N))$ or $m' = \mu(m) \in \mu(N)$. Therefore, $\mu(N)$ is a ϕ-δ-primary hyperideal of $\widehat{S}$.

As an instant consequence of the previous theorem, we get the following explicit results. □

Corollary 4

(i) *Let N be a ϕ-δ-primary hyperideal of $\mathfrak{R}$ and M be a hyperideal of $\mathfrak{R}$ with $M \nsubseteq N$. Suppose that ϕ is global reduction function and δ is global expansion function. Then $N \cap M$ is a ϕ-δ-primary hyperideal of M.*

(ii) *Let N and M be two hyperideals of $\mathfrak{R}$ with $M \subseteq N$. Suppose that ϕ is global reduction function and δ is a global expansion function. Then N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$ if and only if N/M is a ϕ-δ-primary hyperideal of $\mathfrak{R}/M$.*

Theorem 11. *Let N be a hyperideal of $\mathfrak{R}$. Then N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$ if and only if $N/\phi(N)$ is a weakly δ-primary hyperideal of $\mathfrak{R}/\phi(N)$.*

Proof. Assume that N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$. Let $(a\oplus\phi(N))\circ(m\oplus\phi(N)) \subseteq N/\phi(N) - \{0_{\mathfrak{R}/\phi(N)}\}$. Then we have $a^\circ m \in N - \phi(N)$. Since N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$, then we get $a \in \delta(N)$ or $m \in N$, which implies $a\oplus\phi(N) \in \delta(N/\phi(N))$ or $m\oplus\phi(N) \subseteq N/\phi(N)$. Hence, $N/\phi(N)$ is a weakly δ-primary hyperideal of $\mathfrak{R}/\phi(N)$. For the converse, let $a, m \in \mathfrak{R}$ such that $a^\circ m \in N - \phi(N)$. This implies that $(a\oplus\phi(N))\circ(m\oplus\phi(N)) \subseteq N/\phi(N) - \{0_{\mathfrak{R}/\phi(N)}\}$. Since $N/\phi(N)$ is a weakly δ-primary hyperideal of $\mathfrak{R}/\phi(N)$, then we get $a\oplus\phi(N) \in \delta(N/\phi(N))$ or $m\oplus\phi(N) \subseteq N/\phi(N)$ implying $a \in \delta(N)$ or $m \in N$. Therefore, N is a ϕ-δ -primary hyperideal of $\mathfrak{R}$.

Let $\mathfrak{R}$ be a commutative Krasner hyperring, $S \subseteq \mathfrak{R}$ be a multiplicatively closed subset of $\mathfrak{R}$ and T be a proper hyperideal of $\mathfrak{R}$. □

Proposition 9. *Let $\phi_S\colon L(\mathfrak{R}_S) \longrightarrow L(\mathfrak{R}_S) \cup \{\varnothing\}$ be a hyperideal reduction function and $\delta_S\colon L(\mathfrak{R}_S) \longrightarrow L(\mathfrak{R}_S)$ be a hyperideal expansion function such that $\delta_S(N_S) = \delta(N)_S$, for each $N \in L(\mathfrak{R})$. If T is a ϕ-δ-primary hyperideal such that $T \cap S = \varnothing$ and $\phi(T)_S \subseteq \phi_S(T_S)$, then T_S is a ϕ_S-δ_S-primary of $\mathfrak{R}_S$.*

Proof. Let $a/s \circ m/t \in T_S - \phi_S(T_S)$, for some $a, m \in \mathfrak{R}; s, t \in S$. Then we have $p^\circ a^\circ m \in T - \phi(T)$, for some $p \in S$, since $\phi(T)_S \subseteq \phi_S(T_S)$. Since T is a ϕ-δ-primary hyperideal of $\mathfrak{R}$, then we have either $a \in \delta(T)$ or $p^\circ m \in T$. If $a \in \delta(T)$, then $a/s \in \delta(T)_S = \delta_S(T_S)$. If $p^\circ m \in T$, then we have $m/t = p^\circ m/p^\circ t \in T_S$. Therefore, T_S is a ϕ_S-δ_S-primary hyperideal of $\mathfrak{R}_S$.

Let $\varphi_i\colon L(\mathfrak{R}_i) \longrightarrow L(\mathfrak{R}_i) \cup \{\varnothing\}$ be hyperideal reduction functions and $\gamma_i\colon L(\mathfrak{R}_i) \longrightarrow L(\mathfrak{R}_i)$ be hyperideal expansion functions for $i = 1, 2$. Suppose that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$. Also, each hyperideal N of $\mathfrak{R}$ has the form $N = N_1 \times N_2$, where N_i is a hyperideal of $\mathfrak{R}_i$. Furthermore, $\phi\colon L(\mathfrak{R}) \longrightarrow L(\mathfrak{R}) \cup \{\varnothing\}$ defined by $\phi(N_1 \times N_2) = \varphi_1(N_1) \times \varphi_2(N_2)$ is a hyperideal reduction function and also the function $\delta\colon L(\mathfrak{R}) \longrightarrow L(\mathfrak{R})$ defined by $\delta(N_1 \times N_2) = \gamma_1(N_1) \times \gamma_2(N_2)$ is a hyperideal expansion function. □

Proposition 10. *Let the notation be as in the above pharagraph and $N = N_1 \times N_2$. Then each of the types of N are ϕ-δ-primary hyperideals of $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$.*

(i) *$N = N_1 \times N_2$, where N_i is a proper hyperideal of $\mathfrak{R}_i$ with $\varphi_i(N_i) = N_i$.*

(ii) *$N = N_1 \times \mathfrak{R}_2$, where N_1 is γ_1-primary hyperideal of $\mathfrak{R}_1$.*

(iii) *$N = \mathfrak{R}_1 \times N_2$, where N_2 is γ_2-primary hyperideal of $\mathfrak{R}_2$.*

(iv) *$N = N_1 \times \mathfrak{R}_2$, where N_1 is φ_1-γ_1-primary hyperideal of $\mathfrak{R}_1$ and $\varphi_2(\mathfrak{R}_2) = \mathfrak{R}_2$.*

(v) *$N = \mathfrak{R}_1 \times N_2$, where N_2 is φ_2-γ_2-primary hyperideal of $\mathfrak{R}_2$ and $\varphi_1(\mathfrak{R}_1) = \mathfrak{R}_1$..*

Proof

(i) Obviously N is ϕ-δ-primary hyperideal, since $N_1 \times N_2 - \phi(N_1 \times N_2) = \oslash$.

(ii) Let $N = N_1 \times \mathfrak{R}_2$, where N_1 is γ_1-primary hyperideal of $\mathfrak{R}_1$. We can see that $N_1 \times \mathfrak{R}_2$ is δ -primary hyperideal of $\mathfrak{R}_1$. By the Proposition 7, N is ϕ -δ-primary hyperideal of $\mathfrak{R}$.

(iii) The proof is similar to (ii).

(iv) Suppose that $(a_1, a_2)\circ(m_1, m_2) \in N_1 \times \mathfrak{R}_2 - \phi(N_1 \times \mathfrak{R}_2) = \varphi_1(N_1) \times \varphi_2(\mathfrak{R}_2)$. Then $(a_1\circ m_1, a_2\circ m_2) \in (N_1 - \varphi_1(N_1)) \times (\mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2))$. Since N_1 is φ_1-γ_1-primary hyperideal of $\mathfrak{R}_1$, then we get either $a_1 \in N_1$ or $m_1 \in \gamma_1(N_1)$. So $(a_1, a_2) \in N_1 \times \mathfrak{R}_2$ or $(m_1, m_2) \in \gamma_1(N_1) \times \mathfrak{R}_2 \subseteq \gamma_1(N_1) \times \gamma_2(\mathfrak{R}_2)$. It means N is a ϕ-δ-primary hyperideal of $\mathfrak{R}$.

(v) The proof is similar to (iv). □

Theorem 12. *Let the notation be as in the Proposition 10, $N = N_1 \times N_2$, where $\varphi_2(N_i) \neq N_i$. Then N is a φ, δprimary hyperideal of $\mathfrak{R}$ if and only if N is one of the following types:*

(i) *$N = N_1 \times \mathfrak{R}_2$, where N_1 is φ_1, γ_1-primary hyperideal of $\mathfrak{R}_1$ that should be γ_1-primary if $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}_2$.*

(ii) *$N = \mathfrak{R}_1 \times N_2$, where N_2 is φ_2, γ_2-primary hyperideal of $\mathfrak{R}_2$ that should be γ_2-primary if $\varphi_1(\mathfrak{R}_1) \neq \mathfrak{R}_1$.*

Proof. (⇐) It follows from the Proposition 10.

(⇒) Suppose that N is a φ, δ-primary hyperideal of $\mathfrak{R}$, where $\varphi_i(N_i) \neq N_i$. Let $a \circ m \in N_1 - \varphi_1(N_1)$, for some $a, m \in \mathfrak{R}_1$. Thus $(a, 0) \circ (m, 0) = (a \circ m, 0) \in N - \varphi(N)$. Since N is a φ-δ -primary hyperideal of $\mathfrak{R}$, then $(a, 0) \in N$ or $(m, 0) \in \delta(N)$. So $a \in N_1$ or $m \in \gamma_1(N_1)$. Therefore N_1 is φ_1, γ_1-primary hyperideal of $\mathfrak{R}_1$. Similarly, we can find N_2 is φ_2-γ_2-primary hyperideal of $\mathfrak{R}_2$. We have to show now that $N_1 = \mathfrak{R}_1$ or $N_2 = \mathfrak{R}_2$. Suppose that $N_2 \neq \mathfrak{R}_2$. Let we take $m_1 \in N_1 - \varphi_1(N_1)$, $m_2 \in -\mathfrak{R}_2 - N_2$. Notice that $(1,0) \circ \delta(m_1, m_2) = (m_1, 0) \in N - \varphi(N)$. This implies that $(1,0) \in \delta(N)$, so we find $1 \in \gamma_1(N_1)$. Then $N_1 = \mathfrak{R}_1$. Similarly, one can easily find $N_2 = \mathfrak{R}_2$, if $N_2 \neq \mathfrak{R}_2$. Without loss of generality $N_1 \neq \mathfrak{R}_1$. Let we show now that N_1 is γ_1 primary hyperideal with $\varphi_2(\mathfrak{R}_2) \neq \mathfrak{R}$. For $m \in \mathfrak{R}_2 - \varphi_2(\mathfrak{R}_2)$, let $x \circ m \in N$, for some $x, m \in \mathfrak{R}_1$. We have that $(x,1) \circ (m, m') = (x \circ m, m') \in N - \varphi(N)$. Since N is a φ-δ primary hyperideal ofthen we find $(x,1) \in \delta(N) = \gamma_1(N_1) \times \gamma_2(N_2)$ or $(m, m') \in N$ which implies that $x \in \gamma_1(N_1)$ or $m \in N_1$. Therefore N_1 is a γ_1-primary hyperideal of $\mathfrak{R}_1$ If $\phi(\mathfrak{R}_1) \neq \mathfrak{R}_1$ and $N_1 \neq \mathfrak{R}_1$, then similarly one can prove that N_2 is a γ_2 primary hyperideal of $\mathfrak{R}_2$. □

5. Conclusion

In this paper, generalizations of prime and primary hyperideals were provided using the function ϕ. We introduced ϕ-prime, ϕ primary and ϕ-δ primary hyperideals and several characterizations to classify them were provided. Many properties of ϕ prime, ϕ -primary and ϕ-δ-primary hyperideals under particular cases were investigated. In the future work, one can develop the study of ϕ-δ-primary hyperideals.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Disclosure

A preliminary version of this manuscript was submitted as a preprint in arxiv.org [26].

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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