Optical and Quantum Electronics (2023) 55:293
https://doi.org/10.1007/s11082-022-04494-x

Investigation of some nonlinear physical models: exact and approximate solutions

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Received: 8 November 2022 / Accepted: 13 December 2022 / Published online: 4 February 2023

Abstract

We use the $\left(m+\frac{1}{G'}\right)$-expansion method in reaching the exact solution of the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq, Newel-Whitehead-Segel, and Zeldovich equations. New solutions in form of the kink, complex and singular solutions are reported. On the other hand, the Adomian decomposition method is employed to find approximate solutions to the the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation. The three-dimensional figures and their corresponding contour plots for the reported solutions are drawn. Also, a table is presented for the approximate solutions. The reported results may be useful in studying physical features of various nonlinear mathematical models.

Keywords Nonlinear physical models · $\left(m+\frac{1}{G'}\right)$-expansion method · Adomian decomposition method · Travelling wave solutions

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1 Introduction

Finding solutions to linear and nonlinear equations attract the attention of many scientists from the different field all over the world, these include biomathematics, fluid dynamics, biology, optic fibers, applied mathematics, and big engineering sciences. Given the fact that methodologies, methods, and approaches differ by science, many conditions have proven useful. Several different approaches were explored, such as the Collective variables approach (Alrashed et al. 2022), the Kudryashov method (Alharthi et al. 2022), the inverse scattering method (Novikov et al. 1984), the modified tanh-coth method (Wazzan 2009), the $(m + 1/G')$-expansion method (Ismael et al. 2022; Bulut and Ismael 2022), (G'/G)-expansion method (Khalique and Mhlanga 2018), $(m + G'/G)$-expansion method (Ismael et al. 2020), homogeneous balance method (Fan 2000), sine-Gordon expansion method (Ismael and Bulut 2020; Bulut et al. 2018; Ali et al. 2020), the $\exp(-\phi(\xi))$-expansion method (Kadkhoda and Jafari 2017), extended tanh method (Zayed and Abdel Rahman 2010), Riccati-Bernoulli sub-ordinary differential equation method (Yang et al. 2015), the tanh method (Wazwaz 2007), the Jacobi elliptic function expansion method (Liu et al. 2001; Tarla et al. 2022a, b), the tanh-sech method (Wazwaz 2006), Hirota's method (Ismael et al. 2022, 2022, 2021), and many more computational methods (Liu et al. 2020; Manafian et al. 2020, 2021; Jawad et al. 2010; Günerhan 2021; Srivastava et al. 2020; Günerhan 2020; Günerhan et al. 2020).

A generalized transformation has been used to establish different various methods of exact solutions for the Kadomtsev-Petviashvili-Boussinesq equation of (3+1)-dimensional B-type. These are well-known solutions (Senthilvelan 2001). The B-type Kadomtsev-Petviashvili-Boussinesq equation in (3+1)-dimensions (Liu and Zhang 2020) is defined

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} + u_{tt} = 0. \tag{1}$$

A differentiable function is $u = u(x, y, z, t)$. The KP-Boussinesq (1) is a nonlinear PDE of second order in time t that represents both right and left-going waves, similar to the Boussinesq equation. The integrable KP equation is given by first-order PDEs in time, but the KP-Boussinesq (1) is not. Wazwaz and El-Tantawy (2017) investigated single- and double-soliton solutions.

The Newel-Whitehead-Segel (NWS)equation is an amplitude equation (Ur Rehman et al. 2021),

$$v_t + lv_{xx} + Mv + Nv^3 = 0. \tag{2}$$

Zeldovich Equation is an equation that keeps coming up in combustion theory (Ur Rehman et al. 2021).

$$v_t + lv_{xx} + Mv^2 + Nv^3 = 0. \tag{3}$$

Temperature is expressed by the unknown u, and heat generation by combustion is expressed by the last term on the right-hand side (Gilding and Kersner 2004). The Newell–Whitehead–Segel equations have wide applicability in mechanical and chemical engineering, ecology, biology, and bio-engineering. There are various approaches for constructing traveling wave solutions (Baskonus 2016; Korkmaz 2018; Valls 2017; Elgazery 2020; Seadawy and Ali 2021).

The following will be how the article is shaped: Introduce the $\left(m + 1/G'\right)$ -expansion method in the second episode. In the third section, we refer to the following focuses: the B-type Kadomtsev-Petviashvili-Boussinesq equation in (3+1)-dimensions and the application of the suggested method to the studied equation. Moreover, in Sect. 4 we give the physical interpretations and remarks on the solutions obtained by the Adomian decomposition method. In Sect. 5, the results and discussion about solutions will present. Also, a comprehensive conclusion is given in Sect. 6.

2 Fundamental properties of $\left(m+\frac{1}{G'}\right)$—expansion method

In this portion, the application steps related to the mathematical method will be given below

1. Take the equation given in Eq. (2)
$$\Upsilon\left(u, u_x, u_t, u_{xx}, u_{tt}, \cdots\right) = 0. \tag{4}$$
and
$$u(x,y,z,t) = U(\xi),\ \xi = a_1x + a_2y + a_3z - a_4t, \tag{5}$$
where $a_1, a_2, a_3, a_4 \neq 0$. Eq. (4) can be constructed by substituting Eq. (3) into the Eq. (3), as a result we get
$$N\left(U, U', U'', \cdots\right) = 0, \tag{6}$$
where $U = U(\xi),\ U' = \frac{dU}{d\xi},\ U'' = \frac{d^2U}{d\xi^2}, \cdots$.

2. Considering the linear second order differential equation
$$G'' + \lambda G' + \mu = 0, \tag{7}$$
where λ and μ are constants, $G(\xi)$ is a function represents the solution of Eq. (5). Special solution for Eq. (4) get as below:
$$U(\xi) = a_0 + \sum_{i=-n}^{n} a_i \left(m + \frac{1}{G'(\xi)}\right)^i, \tag{8}$$
where a_i $(i = 0, \pm1, \pm2, \pm3, ..., \pm n)$ are constants and n is a positive integer which is going to be found by balancing procedure in Eq. (4). By taking Eq. (5) and Eq. (6) together and replaced by Eq. (4), the obtained results is an equation of polynomial $\left(m + 1/G'\right)^i,\ (i = 0, 1, 2, ..., n)$.

3. We solve the coefficient of a polynomial with $\left(m + 1/G'\right)^i, (i = 0, \pm1, \pm2, ...)$ and the resulting algebraic system is resolved we will specify the values of a_i $(i = -n, ..., -1, 0, 1, ..., n)$. We obtain the solutions to Eq. (4) with the help of a computer package program and we can categorize the exact solutions to Eq. (4). 3-D graphs representing stationary waves can be drawn by giving free parameters to the obtained traveling wave solutions.

3 B-type Kadomtsev-Petviashvili-Boussinesq equation

To discover solutions to the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation Eq. (1), we used the -expansion method. And then, take traveling wave transformation and perform the transformation

$$u(x,y,z,t)=U(\eta),\eta=b_1x+b_2y+b_3z-b_4t, \tag{9}$$

where $b_1,b_2,b_3,b_4\neq 0$ are non-zero constants. Using this equation into Eq. (1), we acquire ordinary differential equation

$$U''\left(3b_1b_3-b_2b_4+{b_4}^2-6{b_1}^2b_2U'\right)-{b_1}^3b_2U^{(4)}=0. \tag{10}$$

We can apply one time integrating

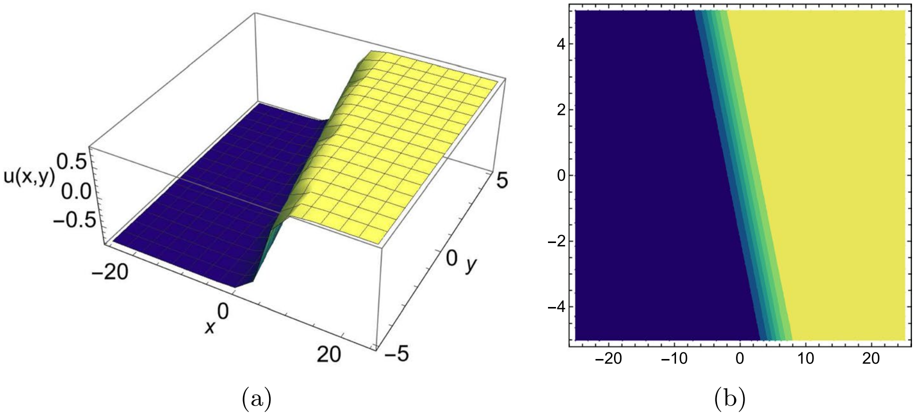
$$\left(3b_1b_3-b_2b_4+{b_4}^2\right)U'-3{b_1}^2b_2\left(U'\right)^2-{b_1}^3b_2U'''=0. \tag{11}$$

According to a relation between $U''U'$ and $U^{(4)}$ in Eq. (8), and we acquire $n=1$. When using a computer package software to solve the algebraic system in Eq. (9), the coefficients and solutions are as follows:

Case 1 When we have $a_{-1}=-2b_1m(\lambda+m\mu)$, $a_1=0$, $b_3=\left(-b_42+b_2\left(b_4\right)\right)\left(\left(+b_13(\lambda+2m\mu)2\right)\right)/\left(3b_1\right)$, we obtain kink soliton solution as seen in Fig. 1.

$$u(x,y,z,t)=\frac{-2a_1m(\lambda+m\mu)}{m+\frac{\lambda+2m\mu}{A_1(\lambda+2m\mu)\mathrm{e}^{(-\lambda-2m\mu)\left(a_1x+a_2y+\frac{\left(a_2\left(a_4+a_1^3(\lambda+2m\mu)^2\right)-a_4^2\right)z}{3a_1}-a_4t\right)}-\mu}}+a_0. \tag{12}$$

Case 2 When we get $a_{-1}=\frac{\sqrt{{b_1}^3b_2\left(3b_1b_3-b_2b_4+{b_4}^2\right)m^2}}{{b_1}^2b_2}-b_1m\lambda$, $a_1=0$, $\mu=-\frac{\sqrt{{b_1}^3b_2\left(3b_1b_3-b_2b_4+{b_4}^2\right)m^2}+{b_1}^3b_2m\lambda}{2{b_1}^3b_2m^2}$, we obtain



(a) (b)

Fig. 1 3-D surfaces of Eq. (12) are plotted in case $\lambda=1,m=1,a_0=1,b_1=1,b_2=1,b_4=1,\delta=1,t=1,z=1,\mu=-0.1$

$$u(x,y,z,t)=\frac{\frac{G}{{b_1}^2b_2}-b_1m\lambda}{m-\frac{2}{\frac{1}{m}+\frac{{b_1}^3b_2\lambda}{G}-2A_1\mathrm{e}^{\frac{(b_1x+b_2y+b_3z-b_4t)G}{{b_1}^3b_2m}}}}+a_0, \tag{13}$$

provided that $G=\sqrt{{b_1}^3b_2\left(3b_1b_3-b_2b_4+{b_4}^2\right)m^2}$. This is singular solution as presented in Fig. 2.

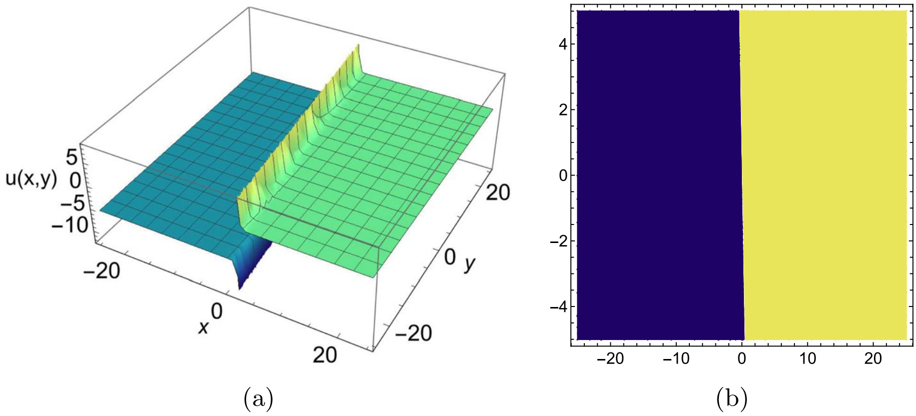
Case 3 In case $a_{-1}=\frac{m(\lambda+m\mu)\left(2\left(1-\mathrm{i}\sqrt{3}\right)b_2b_3(\lambda+2m\mu)^2+2^{1/3}\left(1+\mathrm{i}\sqrt{3}\right)(G_2)^{2/3}\right)}{2^{2/3}b_2(\lambda+2m\mu)^2(G_2)^{1/3}}$, $a_1=0$, $b_1=\frac{\mathrm{i}\left(2\left(\mathrm{i}+\sqrt{3}\right)b_3-\frac{2^{1/3}\left(\sqrt{3}-\mathrm{i}\right)(-G_2)^{2/3}}{b_2(\lambda+2m\mu)^2}\right)}{22^{2/3}(G_2)^{1/3}}$, we obtain

$$u(x,y,z,t)=\frac{m(\lambda+m\mu)\left(2^{1/3}\left(1+\mathrm{i}\sqrt{3}\right){G_2}^{2/3}+2\left(1-\mathrm{i}\sqrt{3}\right)b_2b_3(\lambda+2m\mu)^2\right)}{2^{2/3}b_2{G_2}^{1/3}(\lambda+2m\mu)^2\left(m+\frac{\lambda+2m\mu}{A_1R_1(\lambda+2m\mu)-\mu}\right)}+a_0, \tag{14}$$

provided that $R_1=\mathrm{e}^{(-\lambda-2m\mu)\left(b_2y+b_3z+\frac{\mathrm{i}x\left(2\left(\mathrm{i}+\sqrt{3}\right)b_3-\frac{2^{1/3}\left(\sqrt{3}-\mathrm{i}\right)(-G_2)^{2/3}}{b_2(\lambda+2m\mu)^2}\right)}{22^{2/3}{G_2}^{1/3}}-b_4t\right)}$ and $G_2=\sqrt{{b_2}^3(\lambda+2m\mu)^6\left(b_2\left(b_2-b_4\right)^2{b_4}^2(\lambda+2m\mu)^2-4{b_3}^3\right)}-{b_2}^2\left(b_2-b_4\right)b_4(\lambda+2m\mu)^4$.

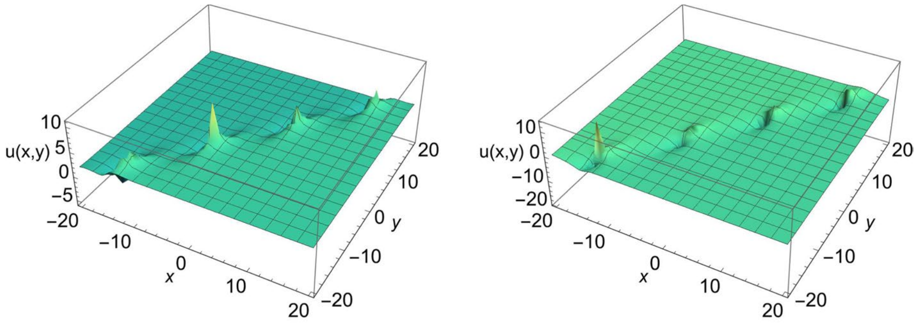
As seen in Fig. 3, a presented solution is a singular.

Case 4 In case $b_1=\frac{\frac{\mathrm{i}\left(\mathrm{i}+\sqrt{3}\right)\left(2{b_2}^2b_4(b_4-b_2)m^4\mu^4+G_3\right)^{2/3}}{b_2m^2\mu^2}-\left(1+\mathrm{i}\sqrt{3}\right)b_3}{8\left(2{b_2}^2b_4\left(b_4-b_2\right)m^4\mu^4+G_3\right)^{1/3}}$, $a_{-1}=\frac{\left(1+\mathrm{i}\sqrt{3}\right)b_3m^2\mu}{4\left(2{b_2}^2b_4\left(b_4-b_2\right)m^4\mu^4+G_3\right)^{1/3}}+\frac{\left(1-\mathrm{i}\sqrt{3}\right)\left(2{b_2}^2b_4\left(b_4-b_2\right)m^4\mu^4+G_3\right)^{1/3}}{4b_2\mu}$, $a_1=\frac{\left(1+\mathrm{i}\sqrt{3}\right)b_3\mu}{4\left(2{b_2}^2b_4\left(b_4-b_2\right)m^4\mu^4+G_3\right)^{1/3}}+\frac{\left(1-\mathrm{i}\sqrt{3}\right)\left(2{b_2}^2b_4\left(b_4-b_2\right)m^4\mu^4+G_3\right)^{1/3}}{4b_2m^2\mu}$, and $\lambda=0$, we obtain



(a) (b)

Fig. 2 3-D surfaces of Eq. (13) are plotted in case $\lambda=1, m=1, \mu=1, A_1=1, a_0=1, b_1=1, b_2=1, b_4=1, t=1, z=1, b_3=1$



(a) Real Part (b) Imaginary Part

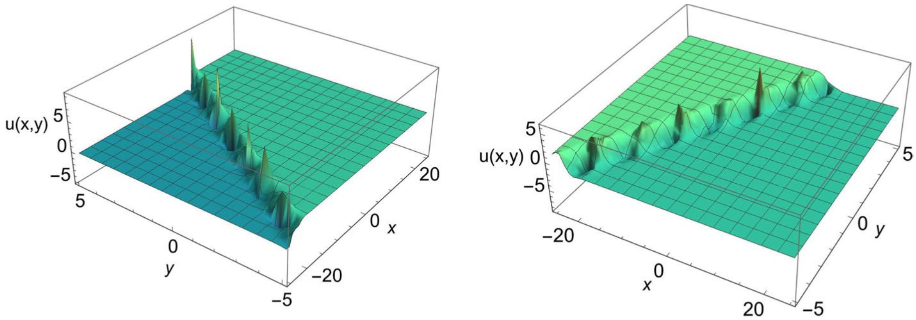
Fig. 3 3-D surfaces of Eq. (14) are plotted in case $\lambda = 1, m = 1, \mu = 1, A_1 = 1, a_0 = 1, b_1 = 1, b_2 = 1, b_4 = 1, t = 1, z = 1, b_3 = 1$

$$u(x,y,z,t) = \frac{R_2\left(m + \frac{\lambda - R_2}{\frac{R_2}{2m} + A_1 e^{(b_1 x + b_2 y + b_3 z - b_4 t)(-\lambda + R)}\left(\lambda - R_2\right)}\right)}{{b_1}^2 b_2 m^2} + a_0. \tag{15}$$

This is complex solution to the studied equation and presented graphically in Fig. 4. Provided that $G_3 = \sqrt{{b_2}^3 m^6 \mu^6 \left(4b_2\left(b_2 - b_4\right)^2 {b_4}^2 m^2 \mu^2 - {b_3}^3\right)}$ and $R_2 = \frac{\sqrt{{b_1}^3 b_2\left(3b_1 b_3 - b_2 b_4 + {b_4}^2\right)m^2} + {b_1}^3 b_2 m \lambda}{{b_1}^3 b_2 m}$.

4 Newel-Whitehead-Segel (NWS) equation

In this part, we tackle to discover solutions to the NWS Equation of Eq. (1) with $\left(m + \left(1/G'\right)\right)$-expansion method. Take traveling wave transformation and perform the transformation



(a) Real Part (b) Imaginary Part

Fig. 4 3-D surfaces of Eq. (15) are plotted in case $\lambda = 1, m = 1, \mu = 1, A_1 = 1, a_0 = 1, b_1 = 1, b_2 = 1, b_4 = 1, t = 1, z = 1, b_3 = 1$

$$u = U(\xi), \xi = \alpha(x - kt), \tag{16}$$

where $\alpha, k \neq 0$ is constant. Putting Eq. (17) into Eq. (2) we acquire ordinary differential equation;

$$MU + NU^3 + \alpha\left(l\alpha U'' - kU'\right) = 0. \tag{17}$$

By taking the relation between U'' and U^3 in Eq. (17), and we acquire $n = 1$. With the help of the package program to gain solutions and values in Eq. (2) follows:

Case 4.1 When $a_0 = \frac{4k\alpha\mu a_{-1}}{2k\alpha\lambda - 3M}$, $a_1 = 0$, $m = -\frac{3M+2k\alpha\lambda}{4k\alpha\mu}$, $N = -\frac{\left(9M^2-4k^2\alpha^2\lambda^2\right)^2}{576k^2M\alpha^2\mu^2a_{-1}^2}$, $l = \frac{2k^2}{9M}$, we obtain

$$u(x,t) = \frac{72A_1 e^{\frac{3M(x-kt)}{2k}} kM^2\alpha\mu a_{-1}}{(2k\alpha\lambda - 3M)\left(3A_1 e^{\frac{3M(x-kt)}{2k}} M(3M + 2k\alpha\lambda) + 2k\alpha(2k\alpha\lambda - 3M)\mu\right)}. \tag{18}$$

We obtain a king solution as seen in Fig. 5.

Case 4.2 In the case $a_0 = \frac{(2k\alpha\lambda-3M)a_1}{4k\alpha\mu}$, $a_{-1} = 0$, $m = -\frac{3M+2k\alpha\lambda}{4k\alpha\mu}$, $N = -\frac{4k^2\alpha^2\mu^2}{9Ma_1^2}$,$l = \frac{2k^2}{9M}$, we obtain

$$u(x,t) = \left(\frac{\lambda - \frac{3M+2k\alpha\lambda}{2k\alpha}}{A_1 e^{(x-kt)\alpha\left(\frac{3M+2k\alpha\lambda}{2k\alpha} - \lambda\right)}\left(\lambda - \frac{3M+2k\alpha\lambda}{2k\alpha}\right) - \mu} - \frac{3M + 2k\alpha\lambda}{4k\alpha\mu}\right)a_1 + \frac{(-3M + 2k\alpha\lambda)a_1}{4k\alpha\mu}. \tag{19}$$

This solution is a kink-soliton solution as seen in Fig. (6).

Case 4.3 When $a_1 = \frac{\frac{i\sqrt{2k}}{\sqrt{Nl}} - 3a_0}{3m}$, $a_{-1} = 0, M = \frac{2k^2}{9l}$, $\lambda = m\mu\left(\frac{2k}{2k+3i\sqrt{2lN}a_0} - 2\right)$, $\alpha = -\frac{2k+3i\sqrt{2lN}a_0}{6lm\mu}$, we obtain

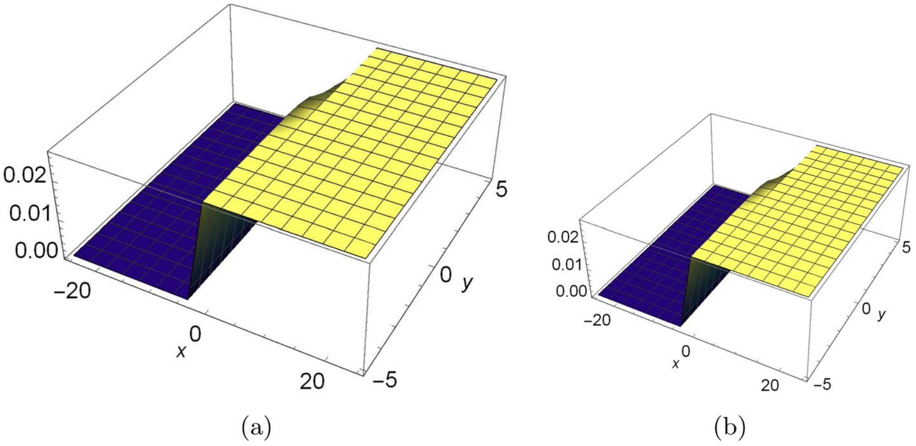
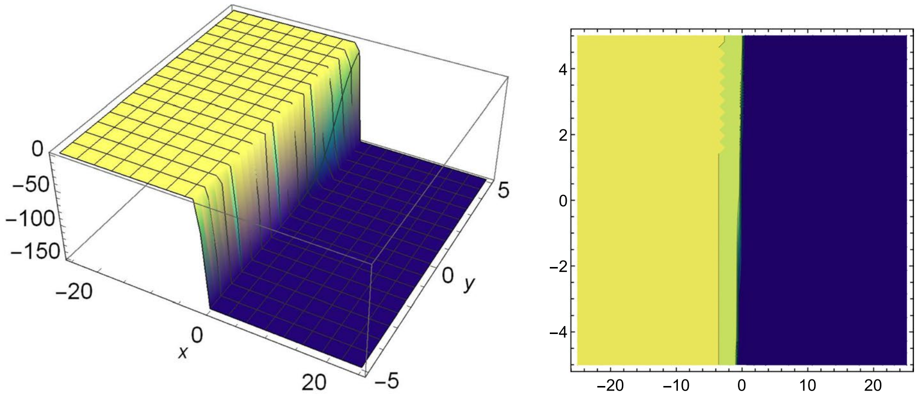


Fig. 5 3-D surfaces of Eq. (18) are plotted in case $\lambda = 1, a_{-1} = 1, M = 1, A_1 = 1, \alpha = 1, k = 0.1, \mu = -0.1$

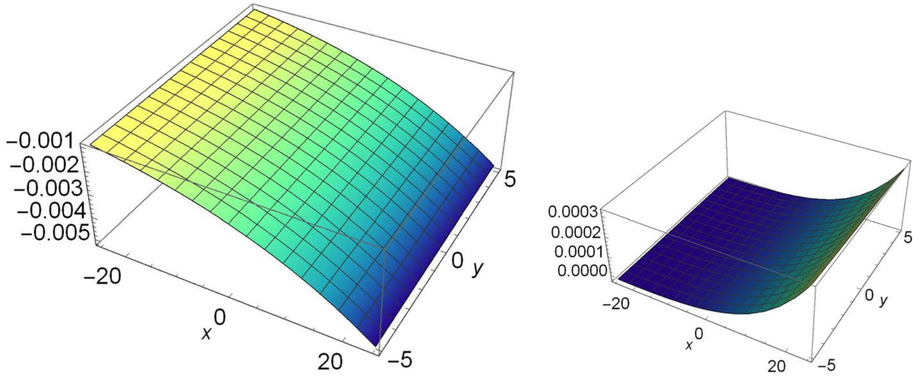


(a) (b)

Fig. 6 3-D surfaces of Eq. (19) are plotted in case $\lambda = 1, a_1 = 1, M = 1, A_1 = 1, \alpha = 1, k = 0.1, \mu = 0.1,$

$$u(x,t) = a_0 + \frac{1}{3m}\left(\frac{\mathrm{i}\sqrt{2}k}{\sqrt{lN}} - 3a_0\right) \left(\left(m + \left(2m\mu + m\mu\left(\frac{2k}{2k + 3\mathrm{i}a_0\sqrt{2lN}} - 2\right)\right)\right) \Big/ \left(A_1 \mathrm{e}^{-\frac{(x-kt)\left(2k+3\mathrm{i}\sqrt{2lN}a_0\right)\left(-2m\mu - m\mu\left(\frac{2k}{2k+3\mathrm{i}a_0\sqrt{2lN}} - 2\right)\right)}{6lm\mu}} \left(2m\mu + m\mu\left(\frac{2k}{2k + 3\mathrm{i}a_0\sqrt{2lN}} - 2\right)\right) - \mu \right) \right). \tag{20}$$

This is complex solution and presented graphically in Fig. 7.



(a) Real Part (b) Imaginary Part

Fig. 7 3-D surfaces of Eq. (24) are plotted in case $\lambda = 1, m = 1, \mu = 0.1, A_1 = 1, \alpha = 1, N = 1, l = 1, a_0 = 1$

5 Zeldovich equation

In this section, we handle to find exact solutions to the Zeldovich Equation of Eq. (2) with $\left(m + \left(1/G'\right)\right)$-expansion method. Take traveling wave transformation and perform the transformation

$$U = u(\xi), \xi = \alpha(x - kt), \tag{21}$$

where $\alpha, k \neq 0$ is constant. Using Eq. (21) in Eq. (3) we acquire ordinary differential equation;

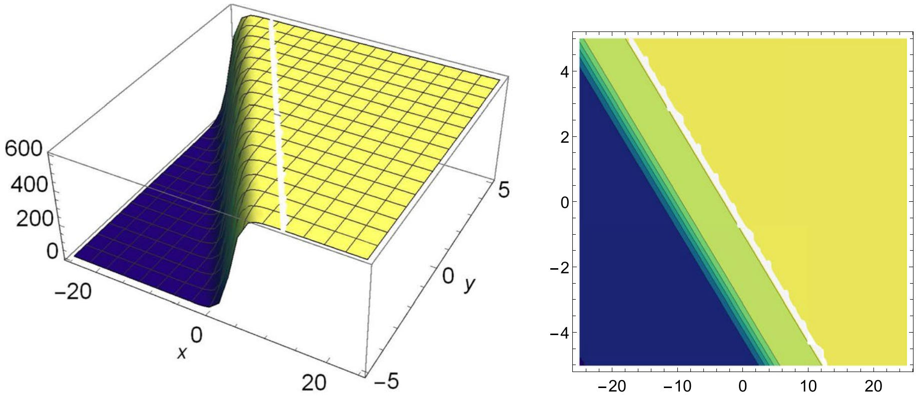
$$MU^2 + NU^3 + \alpha\left(l\alpha U'' - kU'\right) = 0. \tag{22}$$

By taking the relation between U'' and U^3 in Eq. (22), and we acquire $m = 1$. With the help of the package program to gain solutions and values in Eq. (3) follows:

Case 5.1 In case $a_0 = M^{-1}k\left(\alpha\lambda - \frac{k\sqrt{M\alpha a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)^2}}{\sqrt{k^3 M a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)}}\right)$, $a_1 = 0$, $m = -\frac{k\alpha\lambda + \sqrt{k\alpha\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)}}{2k\alpha\mu}$, $N = \frac{M^2\sqrt{k^3 M a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)}}{2k^2\sqrt{M\alpha a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)^2}}$, $l = -\frac{\sqrt{k^3 M a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)}}{\sqrt{M\alpha a_{-1}\left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)^2}}$, we get a singular solution as seen in Fig. 8.

$$u(x,t) = \frac{k\left(\alpha\lambda - \frac{k\sqrt{M\alpha a_{-1}{G_4}^2}}{\sqrt{k^3 M a_{-1} G_4}}\right)}{M} + \frac{2a_{-1}}{\sqrt{G_4}\left(\frac{1}{k\alpha\mu} + \frac{2}{A_1\sqrt{k\alpha G_4}\,\mathrm{e}^{\frac{(kt-x)\sqrt{k\alpha G_4}}{k}} - k\alpha\mu}\right) - \frac{\lambda}{\mu}}, \tag{23}$$

provided that $G_4 = \left(k\alpha\lambda^2 - 2M\mu a_{-1}\right)$.



(a) (b)

Fig. 8 3-D surfaces of Eq. (23) are plotted in case $\lambda = 1, a_{-1} = 1, M = -0.01, A_1 = 1, \alpha = 1, k = -3, \mu = -0.1$

Case 5.2 In case $a_0 = \frac{\left(\alpha\lambda - \sqrt{\alpha^2(\lambda+2m\mu)^2}\right)a_1}{2\alpha\mu}$, $a_{-1} = 0$, $M = -\frac{2k\alpha\mu}{a_1}$, $N = -\frac{2k\alpha^2\mu^2}{\sqrt{\alpha^2(\lambda+2m\mu)^2 a_1^2}}$, $l = \frac{k}{\sqrt{\alpha^2(\lambda+2m\mu)^2}}$, we have

$$u(x,t) = \frac{a_1}{2}\left(\frac{\alpha\lambda - \sqrt{\alpha^2(\lambda+2m\mu)^2}}{\alpha\mu} + \left(m + \frac{\lambda + 2m\mu}{A_1 \mathrm{e}^{(kt-x)\alpha(\lambda+2m\mu)}(\lambda+2m\mu) - \mu} \right)\right). \tag{24}$$

This is kink solution as shown in Fig. 9.

Case 5.3 When we have

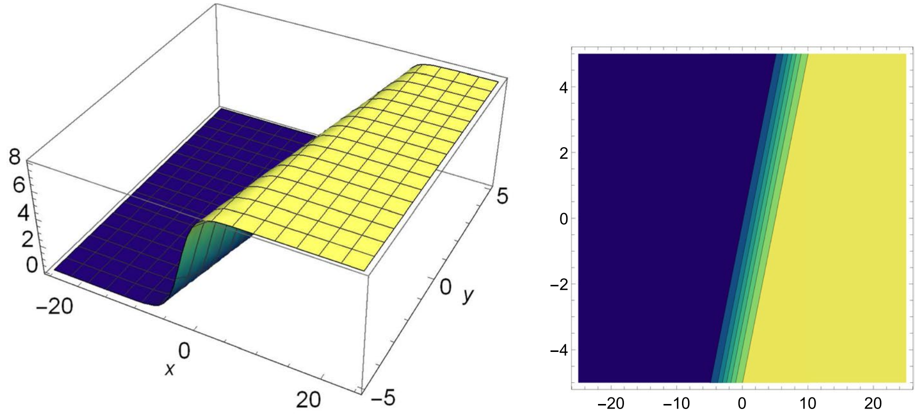
$$\begin{aligned} a_0 &= \frac{2lm\alpha^2\mu(\lambda+2m\mu)}{M}, a_1 = 0, a_{-1} = \frac{2lm\alpha^2(\lambda+m\mu)(\lambda+2m\mu)}{M}, \\ k &= -l\alpha(\lambda+2m\mu), N = -\frac{M^2}{2l\alpha^2(\lambda+2m\mu)^2}, \end{aligned} \tag{25}$$

a kink-soliton solution (see Fig. 10) is

$$u(x,t) = \frac{2A_1 lm\alpha^2(\lambda+2m\mu)^3}{M\left(\mathrm{e}^{\alpha(\lambda+2m\mu)(x+lt\alpha(\lambda+2m\mu))}(\lambda+m\mu) + A_1 m(\lambda+2m\mu)\right)}. \tag{26}$$

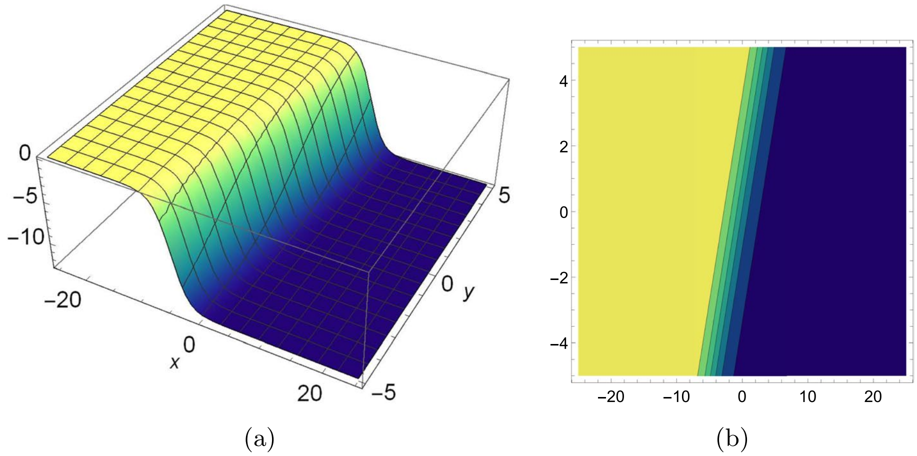
Case 5.4 When we have $a_0 = \frac{\lambda a_1}{4\mu}$, $a_{-1} = -\frac{\lambda^2 a_1}{32\mu^2}$, $M = -\frac{3l\alpha^2\lambda\mu}{2a_1}$, $k = 0$, $m = \frac{\left(3\sqrt{2}-4\right)\lambda}{8\mu}$, $N = -\frac{2l\alpha^2\mu^2}{a_1^2}$, a solution is

$$u(x,t) = \frac{3\lambda\left(9\left(3\sqrt{2}-4\right)A_1^2\lambda^2 - 24\sqrt{2}A_1\mathrm{e}^{\frac{3x\alpha\lambda}{2\sqrt{2}}}\lambda\mu - 8\left(4+3\sqrt{2}\right)\mathrm{e}^{\frac{3x\alpha\lambda}{\sqrt{2}}}\mu^2\right)a_1}{4\mu\left(4\mathrm{e}^{\frac{3x\alpha\lambda}{2\sqrt{2}}}\mu - 3\sqrt{2}A_1\lambda\right)\left(\left(9-6\sqrt{2}\right)A_1\lambda + 2\left(4+3\sqrt{2}\right)\mathrm{e}^{\frac{3x\alpha\lambda}{2\sqrt{2}}}\mu\right)}. \tag{27}$$



(a) (b)

Fig. 9 3-D surfaces of Eq. (24) are plotted in case $\lambda = 1, a_1 = 1, M = -0.1, A_1 = 1, \alpha = 1, k = 1, \mu = -0.1, m = 1$



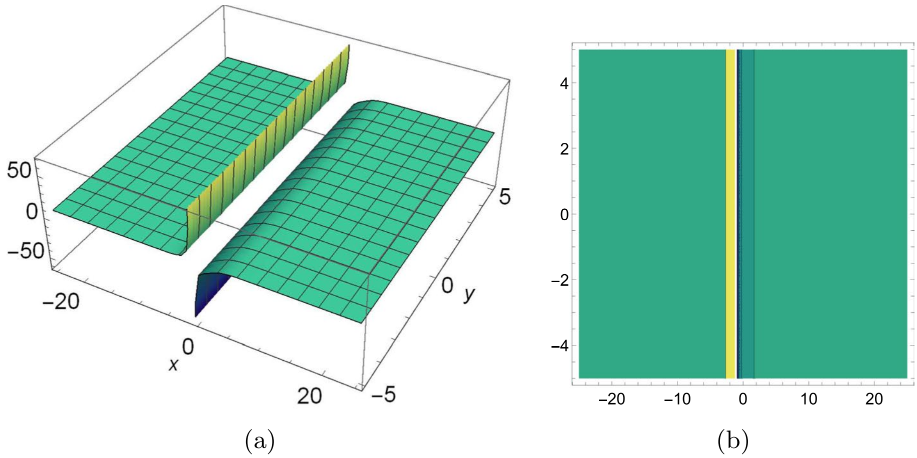
(a) (b)

Fig. 10 3-D surfaces of Eq. (26) are plotted in case $\lambda = 1, a_1 = 1, M = -0.1, A_1 = 2, \alpha = -1, \alpha = -1, k = -0.1, \mu = -0.1, m = 1, l = 1$

This singular solution is presented in Fig. 11.

6 Numerical solution

In this section, we solve the suggested equation numerically. Different techniques have been used to study numerical solutions to PDE such as the penalty method (Shimizu and Aiyoshi 1981), the Laplace-based method (Nuruddeen 2022), a two-step modified Natural decomposition method (Nuruddeen et al. 2018), the meshless method (Aziz 2015), the Adomian decomposition method (Ismail et al. 2004), the Haar wavelet method (Jena and Chakraverty 2019), the Fibonacci matrix polynomial method (Düçünceli and Çelik 2017), the DTM-Pade method (Yousif et al. 2017) and many other techniques. In this work, we



(a) (b)

Fig. 11 3-D surfaces of Eq. (27) are plotted in case $\lambda = 1, a_1 = 1, M = 1, A_1 = 1, \alpha = 1, k = 0.1, \mu = -0$

use the Adomian decomposition method to analyze the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation. Take the following rewrite of Eq. (1):

$$Lu = \left(-u_{ty} + u_{xxy} + 3u_{xx}u_y + 3u_x u_{yx} - 3u_{xz}\right), \tag{28}$$

where $L = \frac{\partial^2}{\partial t^2}$. From Eq. (12) when $b_1 = 1, b_2 = -1, b_4 = 1, A_1 = 1, \lambda = 0.1, \mu = 1, b_0 = 0$ we can find the initial conditions that cover Eq. (15) as follows

$$u(x,y,z,0) = \frac{-2b_1 m(\lambda + m\mu)}{m + \frac{\lambda+2m\mu}{A_1(\lambda+2m\mu)\mathrm{e}^{(-\lambda-2m\mu)\left(b_1 x + b_2 y + \frac{\left(b_2\left(b_4 + b_1^3(\lambda+2m\mu)^2\right) - b_4^2\right)z}{3b_1}\right)} - \mu}} + a_0. \tag{29}$$

Defining the inverse operator $L^{-1}(*) = \int_0^t \int_0^t (*)dtdt$ and using it on Eq. (1), we get

$$u = \int_0^t \int_0^t \left(-u_{ty} + u_{xxy} + 3u_x u_{yx} + 3u_{xx}u_y - 3u_{xz}\right)dtdt, \tag{30}$$

where the nonlinear terms of Eq. 28 are given by

$$u_x u_{yx} = \sum_{k=0}^{\infty} A_k,\ u_{xx}u_y = \sum_{k=0}^{\infty} B_k. \tag{31}$$

Adopting the set of rules for the Adomian polynomials proposed via way of means of Adomian Adomian (1994), we defined:

$$A_i = \sum_{k=0}^{i} \left(\frac{\partial u}{\partial x}\right)_k \left(\frac{\partial^2 u}{\partial y \partial x}\right)_{i-k},\ B_i = \sum_{k=0}^{i} \left(\frac{\partial^2 u}{\partial x^2}\right)_k \left(\frac{\partial u}{\partial y}\right)_{i-k},\ \forall i = 0, 1, ..., n. \tag{32}$$

We note that A_i, B_i are the Adomian polynomial of u_k. The obtained results are investigated graphically and numerically as shown in Fig. 12 and Table 1.

7 Results and discussion

In this work, the $(m + 1/G')$-expansion and the Adomian decomposition methods are used to explore a class of exact solutions to the studied equation and numerical solutions to the studied equation, respectively. Kink-type solutions and singular solutions are presented via $(m + 1/G')$-expansion method. The Adomian decomposition method produced numerical results that provided approximate solutions to the exact solution for this equation that we are considering. As it can be understood from the approximate solutions given in detail in Table 1 and Fig. 12, the approximate solutions obtained are very close to each other.

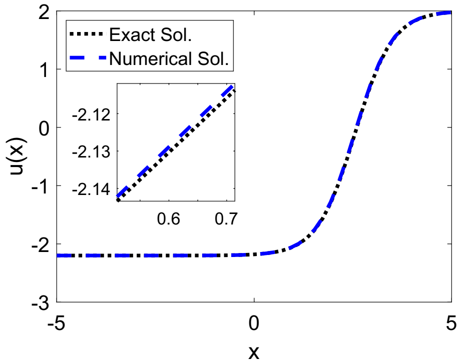


Fig. 12 2D graph of exact solutions under Eq. (12) and corresponding numerical solution plotted when $t = 0.01, y = 0.1, z = 1, b_1 = 1, b_2 = -1, b_4 = 1, A_1 = 1, \lambda = 0.1, \mu = 1, a_0 = 0$

Table 1 Analytical and numerical solutions of u(x,t) for the studied equation and its absolute errors when $y = 0.1, z = 1, b_1 = 1, b_2 = -1, b_4 = 1, A_1 = 1, \lambda = 0.1, \mu = 1, a_0 = 0$

x_i	t_i	Exact solution	Numerical solution	Absolute error
0.01	0.001	−2.1796464	−2.17960379	4.266825×10^{-5}
0.02	0.001	−2.17921665	−2.17917309	4.356477×10^{-5}
0.03	0.001	−2.1787778	−2.17873333	4.447993×10^{-5}
0.04	0.001	−2.1783297	−2.17828434	4.541412×10^{-5}
0.05	0.001	−2.17787230	−2.17782593	4.636771×10^{-5}
0.06	0.001	−2.17740523	−2.17735789	4.734111×10^{-5}
0.07	0.001	−2.17692835	−2.17688002	4.833471×10^{-5}
0.08	0.001	−2.17644147	−2.17639212	4.934893×10^{-5}
0.09	0.001	−2.17594438	−2.17589399	5.038419×10^{-5}

8 Conclusion

This study employed the $\left(m + \frac{1}{G'}\right)$-expansion method in reaching the exact solution of the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq, Newel–Whitehead–Segel, and Zeldovich equations. A soliton, sometimes referred as a solitary wave, is a self-reinforcing wave packet that holds its structure while moving at a fixed speed. Nonlinear and dispersive effects in the medium are suppressed, causing the formation of scattering. (Dispersive effects are a characteristic of some systems where a wave's speed is frequency-dependent.) A widespread family of weakly nonlinear dispersive partial differential equations that characterize physical systems has solutions termed as solitons. New solutions in form of the kink and singular solutions are reported. We also employed the Adomian decomposition method to reach the approximate solutions of the (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation. The $\left(m + \frac{1}{G'}\right)$-expansion method is powerful in reaching analytical solutions to various nonlinear equations. The Adomian decomposition method is an efficient technique for finding approximate solutions to various nonlinear equations. The results presented in this study are helpful in explaining the physical features of various nonlinear equations. We plan to study the extended classical version of the studied nonlinear models in this work.

Author Contributions This work was written with equal contributions from each author. The final manuscript was read and approved by all writers.

Funding Not applicable.

Availability of data and materials Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have neither financial nor conflict interest.

Ethical approval The authors state that this research paper complies with ethical standards. This research paper does not involve either human participants or animals.

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